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O. Neugebauer M. H. Stone O. Veblen
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Mathematical Reviews

Vol. 7, No. 6 JUNE, 1946 Pages 233-272

ALGEBRA

Riordan, John. Three-line latin rectangles. II. Amer. Math. Monthly 53, 18–20 (1946). [MF 15307]

The number of n by 3 Latin rectangles is shown to be $\sum_n C_i D_i D_{n-i} u_{n-2i}$, where D_n is the number of permutations discordant with $(1, \cdots, n)$ and u_n the number also discordant with $(2, \cdots, n, 1)$. This formula is derived from the one in the first paper [Amer. Math. Monthly 51, 450–452 (1944); these Rev. 6, 113]. *I. Kaplansky.*

Jessen, Axel. A combinatorial problem. Mat. Tidsskr. B. 1945, 58 (1945). (Danish) [MF 14261]

The author shows how to arrange four 1's, 2's, $\cdots$, 13's in an array of 4 rows and 13 columns so that each pair i, j $(i \neq j)$ occurs in exactly one column. *I. Kaplansky.*

Husain, Q. M. On the totality of the solutions for the symmetrical incomplete block designs: $\lambda=2$, $k=5$ or 6. Sankhyā 7, 204–208 (1945). [MF 15136]

The author constructs all solutions for the incomplete balanced block designs with parameters $v=b=11$, $r=k=5$, $\lambda=2$ and $v=b=16$, $r=k=6$, $\lambda=2$. There are (isomorphic solutions being considered equal) only one solution of the first and three of the second design. The author also discusses methods of establishing the isomorphism between two designs. *H. B. Mann* (Columbus, Ohio).

Chowla, S. A contribution to the theory of the construction of balanced incomplete block designs. Proc. Lahore Philos. Soc. 7, 3 pp. (1945). [MF 15180]

The author states and sketches a proof of the following theorem. Let k be any positive integer; then, for every sufficiently large prime p such that $p \equiv 1$ (k^2-k), $(p-1)/(k^2-k) \not\equiv 0$ (2), there exists a balanced incomplete block design with parameters $v=p$, $b=p(p-1)/(k^2-k)$, $r=(p-1)/(k-1)$, $\lambda=1$, $k=k$. This result implies that to every pair of integers a, b there exists a balanced block design with $k=a$, $\lambda=b$. *H. B. Mann* (Columbus, Ohio).

Chowla, S. A new case of a "complete l—m—n configuration." Proc. Lahore Philos. Soc. 6, no. 1, 13 (1944). [MF 15094]

The configuration in question is the 12—2—49 configuration of 197 elements. It is obtained from the set of biquadratic residues mod 197 by addition of 0, 1, $\cdots$, 196. *H. B. Mann* (Columbus, Ohio).

Chowla, S. Another case of a "complete l—m—n configuration." Proc. Lahore Philos. Soc. 6, no. 1, 14 (1944). [MF 15095]

The configuration is the 6—2—25 configuration of 101 elements. It is obtained from the set of biquadratic residues mod 101 by addition of 0, 1, $\cdots$, 100. *H. B. Mann.*

Hadwiger, H. Gruppierung mit Nebenbedingungen. Mitt. Verein. Schweiz. Versich.-Math. 43, 113–122 (1943). [MF 14534]

Let $A(n, m, k)$ be the number of ways of partitioning n objects into m sets of at most k each. The author derives the generating function

$$\sum A(n, m, k)x^m y^n/n! = \exp x(y+y^2/2!+\cdots+y^k/k!)$$

and related results. The case $k=n$ is classical [E. Netto, Lehrbuch der Combinatorik, 2d ed., Teubner, Leipzig, 1927, p. 169]. *I. Kaplansky* (Chicago, Ill.).

Claeys, A. Sur quelques déterminants trigonométriques. Mathesis 54, 298–301 (1942). [MF 15527]

Garnea, E. G. Study of some symmetric and homogeneous functions of m variables. Revista Soc. Cubana Ci. Fís. Mat. 2, 1–8 (1945). (Spanish) [MF 14180]

The homogeneous symmetric functions $\sigma_i^{(p)}$ which the author studies arise upon dividing by the Vandermonde determinant D of m variables those determinants obtained from D by suppressing the $(m-i+1)$th row $(i=1, \cdots, m)$ and annexing as mth row the elements $u_1^{m+p-1}, \cdots, u_m^{m+p-1}$, $p \geqq -m+1$. Since the $\sigma_i^{(p)}$ reduce to the elementary symmetric functions S_i of m variables $(i=1, \cdots, m)$ for $p=1$, they may be considered as generalizations of those functions. The paper is concerned with establishing the relation $\sigma_i^{(p)} = S_i\sigma_1^{(p-1)} - \sigma_{i+1}^{(p-1)}$ and applying it to the study of a system of linear equations in $\sigma_1^{(p)}, \cdots, \sigma_m^{(p)}$.
L. M. Blumenthal (Columbia, Mo.).

Harish-Chandra. Algebra of the Dirac-matrices. Proc. Indian Acad. Sci., Sect. A. 22, 30–41 (1945). [MF 13398]

A new derivation of relations between the Dirac-matrices developed originally by W. Pauli [Ann. Inst. H. Poincaré 6, 109–136 (1936)]. *E. P. Wigner* (Princeton, N. J.).

Sarymsakov, T. A. Sur les suites des matrices stokhastiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 326–328 (1945). [MF 14018]

Let the infinite matrix $P=(p_{i,k})$ be stochastic, that is, $p_{i,k} \geqq 0$, $\sum_k p_{i,k}=1$, $\sum_i p_{i,k} \neq 0$. The author defines a sequence of matrices Ω_n as follows. Consider an arbitrary sequence of nonnegative numbers q_k with $\sum q_k=1$. The matrix Ω_n is of order $n+1$ and is obtained by bordering the main submatrix of P of order n by the following elements:

$$\overset{*}{p}_{i,n+1} = \sum_{j=n+1}^{\infty} p_{i,j},$$

$$\overset{*}{p}_{n+1,k} = r_n^{-1} \sum_{j=n+1}^{\infty} q_j p_{j,k},$$

$$\overset{*}{p}_{n+1,n+1} = r_n^{-1} \sum_{j=n+1}^{\infty} q_j \overset{*}{p}_{j,n+1};$$

here $r_n = \sum_{j=n+1}^{\infty} q_j$. In case $r_n=0$ the new elements should be interpreted as $\overset{*}{p}_{n+1,k}=0$ if $k<n+1$ and $\overset{*}{p}_{n+1,n+1}=1$.

Characteristic properties of P are deduced from those of Ω_n in seven theorems of which the following are typical.

With a finite number of exceptions either all or none of the Ω_n are of the form

$$\begin{pmatrix} R & Q \\ S & T \end{pmatrix},$$

where the elements are square matrices and either $Q=0$ or $S=0$. If Ω_1 is imprimitive, all Ω_n are imprimitive (have a characteristic root λ with $|\lambda|=1$ in addition to the trivial characteristic root 1). If there is a j with $p_{jj}>0$, all Ω_n are primitive. *W. Feller* (Ithaca, N. Y.).

Williamson, John. Note on the equivalence of nonsingular pencils of Hermitian matrices. Bull. Amer. Math. Soc. **51**, 894–897 (1945). [MF 14455]

The author [Amer. J. Math. **57**, 475–490 (1935)] developed the theory of the equivalence of two nonsingular pencils of Hermitian or symmetric matrices with respect to a commutative field of characteristic 0. Now, using a rational canonical form due to the reviewer [not, as he states, to Jacobson], he extends these results to all fields of characteristic not 2. *C. C. MacDuffee* (Madison, Wis.).

Venkatachaliengar, K. Pairs of symmetric and skew matrices in an arbitrary field. I. Proc. Indian Acad. Sci., Sect. A. **22**, 243–264 (1945). [MF 14451]

The author obtains necessary and sufficient conditions that the pair of symmetric matrices A, B be congruent to the pair C, D, that is, that there exist a nonsingular matrix P such that $PAP'=C$ and $PBP'=D$, for matrices with elements in any field except a field of characteristic 2. This problem, for fields of characteristic zero, was solved by Dickson [Trans. Amer. Math. Soc. **10**, 347–360 (1909)] and later by the reviewer [Amer. J. Math. **57**, 475–490 (1935)], who recently showed that his proofs were also valid for all fields except for those of characteristic 2 [see the preceding review]. The author's treatment is new and complete in itself, including the singular case. He first shows that the pencil $rA+sB$ is congruent to a direct sum $rA_1\dot{+}sB_1\dot{+}rA_2\dot{+}sB_2\dot{+}\cdots\dot{+}rA_m\dot{+}sB_m$, where at least one of A_i or B_i is nonsingular, $i=1, \cdots, m$, and rA_i+sB_i has a single elementary divisor $[p_i(r, s)]^{e_i}$ and if B_i is nonsingular, $p_i(1, s)$ is irreducible. He then determines a canonical form under congruent transformations for the pencil A_i+tB_i by methods analogous to those used by MacDuffee [Vectors and Matrices, Carus Math. Monograph Series, no. 7, Ithaca, N. Y., 1943, pp. 134–136; these Rev. **5**, 30] in finding the canonical form of a matrix under similarity transformations. He then solves his problem in the usual manner by finding necessary and sufficient conditions that two pencils in canonical form be congruent.

J. Williamson (Flushing, N. Y.).

Schur, Issai. Ein Satz ueber quadratische Formen mit komplexen Koeffizienten. Amer. J. Math. **67**, 472–480 (1945). [MF 13935]

Every symmetric matrix C with real or complex elements can be reduced to the diagonal form $UCU'=[\omega_1, \omega_2, \cdots, \omega_n]$, where U is unitary and the ω_i are the real nonnegative square roots of the characteristic roots of $C\bar{C}$. Hence every symmetric bilinear form $f(x, y)$ in the Hilbert space $\sum|x_\nu|^2\leqq 1$, $\sum|y_\nu|^2\leqq 1$ assumes its maximum absolute value on the boundary with $x_\nu=y_\nu$. A symmetric complex quadratic form $\sum_{\kappa,\lambda=1}^{\infty}c_{\kappa\lambda}x_\kappa y_\lambda$ is restricted if and only if the corresponding bilinear form is restricted. *C. C. MacDuffee.*

Bourgin, D. G. Quadratic forms. Bull. Amer. Math. Soc. **51**, 907–908 (1945). [MF 14458]

Let Q be a real indefinite quadratic form in $n+2$ variables. It is shown that the signature of Q is $n+2-\sum_0^n R_j$, where R_j are the mod 2 Betti numbers of the hypersurface $Q=0$.

C. C. MacDuffee (Madison, Wis.).

Faddeyev, D. K. Construction of fields of algebraical numbers whose Galois group is a group of quaternion units. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 390–392 (1945). [MF 14506]

Let Q_{24} denote the group generated by the quaternions ± 1, i, j, k, $\frac{1}{2}(-1+i+j+k)$. The author proves that any cyclic algebraic number field, over an arbitrary ground field, can be embedded in a field with group Q_{24}. To do so one need only find a triplet of numbers $\alpha_1, \alpha_2, \alpha_3$ in the cubic field which are orthogonal to the conjugate triplets (that is, $\alpha_1\alpha_1'+\alpha_2\alpha_2'+\alpha_3\alpha_3'=\alpha_1\alpha_1''+\alpha_2\alpha_2''+\alpha_3\alpha_3''=0$). Such triplets can always be found. The embedding uses the algebra of quaternions over the cubic field. Although Q_{24} is homomorphic to the tetrahedral group of order 12, not every field with the tetrahedral group can be embedded in a field with group Q_{24}. When the ground field is the rational field, the author gives an arithmetic necessary and sufficient condition for this embedding to be possible. *G. Whaples.*

Faddejev, D. K. On a problem of analytical geometry. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 539–540 (1945). [MF 14418]

Given a symmetric 3×3 matrix A, with coefficients in a given field R, the problem is to find all transforms of A, under unitary orthogonal transformations, which also have coefficients in R. If the characteristic equation of A is irreducible, and k the field generated by one of its roots, these transforms of A correspond to triplets of numbers of k orthogonal to conjugate triplets. When k is cyclic, the problem is equivalent to that of the preceding review. When k has the symmetric group, the author describes a construction which gives all these triplets, and hence all transforms.

G. Whaples (Philadelphia, Pa.).

Conway, A. W. Quaternions and matrices. Revista Unión Mat. Argentina **11**, 11–17 (1945). (Spanish) [MF 14098]

Translation of an article in Proc. Roy. Irish Acad. Sect. A. **50**, 98–103 (1945); these Rev. **6**, 199.

***Gonçalves Miranda, Manuel. Associative and modular vectorial multiplications. I.** Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. 11, xix+160 pp. (1945). (Portuguese. French summary)

In a systematic way the author develops the theory of associative vector multiplication defined in terms of bilinear forms in the components of the factors; for sets (a), (b) of n real or complex numbers, the kth element of the product is defined by $(a\cdot b)^k=\alpha_{ij}{}^k a^i b^j$. The topics treated are: Contravariant and covariant multiplication; Associative property; Matrices of representation; Division of vectors; Modular multiplication; Modulus of a vector; Zeros of vectors; Semimodular and nonmodular multiplication; Some commutation conditions; Application to the construction of mixed multiplications. *E. F. Beckenbach* (Los Angeles, Calif.).

Calugareanu, Georges. Sur le calcul symbolique de Cayley-Aronhold-Clebsch dans la théorie des invariants. Mathematica, Timişoara **21**, 95–109 (1945). [MF 13975]

The connexion between differential operators and the Clebsch-Aronhold symbols for binary and higher forms is

well known and dates from the original work of Cayley a century ago. The present work gives an interesting alternative method which is based on complex integration. A binary $n$ic $(A, B, \cdots, K)(x, y)^n$ is equal to

$$(2\pi i)^{-2}\int\int \psi(p, q)(px+qy)^n dp dq,$$

where $\psi(p, q)=(xy)^{-1}(Ax^{-n}+Bx^{-n+1}y^{-1}+\cdots+Ky^{-n})$ and where the integral is taken for each of the independent complex variables p and q along a simple closed contour surrounding the origin once. This function ψ is called the kernel (noyau) and contains the $n+1$ coefficients of the binary form but without their binomial coefficients. The factor $(px+qy)^n$ then behaves like the symbolic expression $(a_1x+a_2y)^n$ for the $n$ic, where the complex numbers p, q now play the role of the classical symbols a_1, a_2.

For the Nary form the necessary kernel is

$$\psi=(x_1x_2\cdots x_N)^{-1}\sum A_{i_1i_2\cdots i_N}x_1^{-i_1}x_2^{-i_2}\cdots x_N^{-i_N}.$$

The method is developed in an illuminating way to illustrate the first fundamental theorem, and several of the simpler invariants and combinants of binary and higher forms, including affine ternary invariants. A technique of nuclear fission (scission du noyau) is applied to the kernel and produces results. *H. W. Turnbull* (St. Andrews).

Motzkin, Théodore. Sur l'équation irréductible $z^n+a_1z^{n-1}+\cdots+a_n=0$, $n>1$, à coefficients complexes entiers, dont toutes les racines sont sur une droite. Les 11 classes de droites admissibles. C. R. Acad. Sci. Paris **221**, 220–222 (1945). [MF 14251]

The author discusses irreducible algebraic equations with complex integers as coefficients all of whose roots are collinear. It is proved that the straight lines which carry these roots fall into eleven classes. Some of these classes only come in question if the degree of the equation is even or divisible by four. The author remarks that this classification can be completely justified by the following fact: for any two straight lines of the same class there is a similitude which transforms every algebraic number of degree n lying on one of the lines into an algebraic number of degree n on the second line. Such a correspondence is not possible for lines of different classes. *O. Todd-Taussky.*

Petterson, Erik L. Eine Darstellung irreduzibler Polynome der imaginärkvadratischen Zahlkörper. Ark. Mat. Astr. Fys. **29A**, no. 15, 11 pp. (1943). [MF 12015]

Let $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial whose coefficients are integers in an imaginary quadratic field K. The author shows that $f(x)$ is irreducible in K if and only if there exists a polynomial $g(x)$, whose coefficients are integers in K, such that $0<|g(\xi)|<1$ for all roots ξ of $f(x)=0$ with the exception of one. It is then shown that a polynomial $f(x)$, irreducible in K, can be put, in infinitely many ways, into the form $f(x)=h(x)+g(x)$, where $h(x)$ and $g(x)$ are relatively prime polynomials whose coefficients are integers in K, such that the degree of $h(x)$ exceeds that of $g(x)$ by unity and $|h(x)|<1$ for all complex numbers x such that $|g(x)|=1$. Conversely, $f(x)$ is irreducible in K whenever it can be put into the form just described. The author notes that the second criterion of irreducibility holds as a sufficient condition also when K is the field of rational numbers, although it is then no longer necessary; it can be shown to be necessary in case a real root of $f(x)=0$ is a vertex of the Newton polygon formed from all the roots of that equation. *H. W. Brinkmann* (Swarthmore, Pa.).

Abstract Algebra

Richardson, M. On weakly ordered systems. Bull. Amer. Math. Soc. **52**, 113–116 (1946). [MF 15450]

A weakly ordered system D is a set with an anti-symmetric relation "$>$"; a solution in the sense of von Neumann and Morgenstern [Theory of Games and Economic Behaviour, Princeton University Press, 1944; these Rev. **6**, 235] is a subset V such that $x\not> y$ for $x, y \epsilon V$ and $x \epsilon D-V$ implies $y>x$ for some $y \epsilon V$. The author shows that a solution exists if all unoriented cycles in the graph of D have an even number of elements. *I. Kaplansky* (Chicago, Ill.).

Schützenberger, Maurice-Paul. Sur certains axiomes de la théorie des structures. C. R. Acad. Sci. Paris **221**, 218–220 (1945). [MF 14250]

One can consider, for every cardinal number α and set U of identities ("laws"), the free lattice with α generators satisfying U; the U form a lattice L under identical implication. It is known that the distributive law U_0 contains every element in L except $x=y$; the author states that there are precisely two sets U_1 and U_2 of laws maximal under U_0. He lists other sets of laws between U_2 and the modular law U_4, including laws which imply the Desargues theorem, and laws which imply that the projective geometries embedded in a modular lattice have the field of integers mod p for coefficients. No proofs are given. *G. Birkhoff.*

Matsuyama, Noboru. On the representation of Boolean algebra. Proc. Imp. Acad. Tokyo **19**, 426–428 (1943). [MF 14837]

The author determines when the space defined by a distributive lattice [H. Wallman, Ann. of Math. (2) **39**, 112–126 (1938)] is a T-space, a T_1-space or a T_2-space; also, when it satisfies the first countability axiom.

G. Birkhoff (Cambridge, Mass.).

Komatu, Atuo. On a characterisation of join homomorphic transformation-lattice. Proc. Imp. Acad. Tokyo **19**, 119–124 (1943). [MF 14801]

Let $J(L)$ be the lattice of all join-homomorphic mappings of a lattice L into the two-element lattice B; $J(L)$ is dually isomorphic to the lattice of ideals of L. Furthermore, (i) $J(L)$ is complete, (ii) all its elements are meets of meet-irreducible elements, (iii) join is continuous in the generalized (o)-topology, (iv) the "finite-meet-reducible" elements a such that $a=\cap\{a_x|X\}$ implies $a=a_{x_1}\cap\cdots\cap a_{x_n}$, for some finite subset of X, form a lattice. Conversely, if L^* satisfies (i)–(iv), then $L^*=J(L)$, where L is dual to the lattice of (iv).

G. Birkhoff (Cambridge, Mass.).

Komatu, Atuo. On a characterisation of order-preserving mapping-lattice. Proc. Imp. Acad. Tokyo **19**, 125–128 (1943). [MF 14802]

Let B^L be the lattice of all order-preserving mappings of a lattice L into the two-element lattice B. The author shows (i) that B^L is complete, (ii) that it is completely distributive, (iii) that every $a\epsilon B^L$ is a join of join-irreducible elements, (iv) that the join-irreducible elements of B^L form a lattice. Conversely, he shows that if L^* satisfies (i)–(iv), and L is the dual of the lattice of join-irreducible elements of L^*, then $L^*=B^L$. *G. Birkhoff* (Cambridge, Mass.).

Cotlar, Mischa. A method of construction of structures and its application to topological spaces and abstract arithmetic. Univ. Nac. Tucumán. Revista A. **4**, 105–157 (1944). (Spanish) [MF 13023]

Let there be given a family Γ of subsets of a "space" E, and a binary relation $x\rho C$ between the $x\epsilon E$ and the $C\epsilon\Gamma$.

Let $C\rho D$ mean $x\rho D$ for all $x\epsilon C$, where $C, D\epsilon\Gamma$. Assume that $x\rho C$ and $C\rho D$ imply $x\rho D$ (transitivity postulate). The author defines an "anagene" as a set $C\epsilon\Gamma$ such that $x\epsilon C$ if and only if $x\rho D$ for all $D\rho C$, and shows that this abstraction includes as special cases the concepts of closed and of measurable sets in Euclidean space, and of closed ideals in partially ordered sets. He shows how to "complete" partially ordered sets using anagenes [cf. H. M. MacNeille, Trans. Amer. Math. Soc. **42**, 416–460 (1937)]. He gives an example of a distributive lattice whose completion by cuts is not modular [Dilworth has given another]. He gives a construction embedding distributive lattices in complete distributive lattices; however, one step of the proof is not clear in the case of uncountable chains of different lengths [a nonconstructive proof is known]. Applications of anagenes are also made to the accessible spaces of Fréchet, and to "arithmetic" partially ordered sets, in which every element has a finite number of predecessors and every two elements have an infinite number of common successors. *G. Birkhoff.*

Funayama, Nenosuke. On the completion by cuts of distributive lattices. Proc. Imp. Acad. Tokyo **20**, 1–2 (1944). [MF 14864]

The author gives an example of a distributive lattice L whose completion by cuts $\bar{L}$ is not modular. [Other examples have recently been given by Dilworth and M. Cotlar; cf. the preceding review.] He also shows that $\bar{L}$ is distributive if and only if, for every normal subset S of L, the lattice-homomorphism $x\rightarrow[\bar{x}\cap S, \bar{x}\cup S]$ is an isomorphism.

G. Birkhoff (Cambridge, Mass.).

Funayama, Nenosuke. On the congruence relations on lattices. Proc. Imp. Acad. Tokyo **18**, 530–531 (1942). [MF 14774]

Let X denote the set of prime quotients of a lattice L of finite dimensions, quasi-ordered by the relation $\geqq$, where $p/q\geqq r/s$ means $u\geqq r>s\geqq v$ for some quotient u/v projective to p/q. The author shows that the lattice of congruence relations on L is isomorphic to the distributive lattice B^X, where B is the two-element Boolean algebra.

G. Birkhoff (Cambridge, Mass.).

Funayama, Nenosuke, and Nakayama, Tadasi. On the distributivity of a lattice of lattice-congruences. Proc. Imp. Acad. Tokyo **18**, 553–554 (1942). [MF 14780]

The authors prove that the congruence relations on any lattice form a distributive lattice satisfying the infinite distributive law $x\cap(\cup y_\alpha)=\cup(x\cap y_\alpha)$, but not necessarily the dual infinite distributive law. (Here $x\geqq y$ means that $a\equiv b \pmod{x}$ if $a\equiv b \pmod{y}$.) *G. Birkhoff.*

Nakayama, Tadasi. Über den Verband der Rechtsideale eines assoziativen Ringes. Proc. Imp. Acad. Tokyo **20**, 664–669 (1944). [MF 14941]

A necessary and sufficient condition is given for a modular lattice to be the lattice of right ideals of an associative ring with unit. The condition is too complicated to be reproduced here. *I. Kaplansky* (Chicago, Ill.).

Nakayama, Tadasi. On Krull's conjecture concerning completely integrally closed integrity domains. I. Proc. Imp. Acad. Tokyo **18**, 185–187 (1942). [MF 14753]

Clifford [Ann. of Math. (2) **41**, 465–473 (1940); these Rev. **2**, 4] raised the question of whether an Archimedean partially ordered Abelian group can be embedded in a vector group with real components. The author gives the following counter-example: the set of continuous functions, finite except on a nowhere dense set, from a Boolean space to the reals and $\pm\infty$. The impossibility of the representation follows from the fact that the intersection of the maximal normal subspaces [G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., vol. 25, New York, 1940, p. 109; these Rev. **1**, 325] is not zero. *I. Kaplansky.*

Nakayama, Tadasi. On Krull's conjecture concerning completely integrally closed integrity domains. II. Proc. Imp. Acad. Tokyo **18**, 233–236 (1942). [MF 14758]

A question related to that in the preceding review was raised by Krull [Math. Z. **41**, 665–679 (1936)]; he conjectured that a completely integrally closed integral domain is an intersection of Archimedean valuation rings. The author obtains a counter-example by taking formal power series with exponents in the group of his preceding example. At the same time he answers another of Krull's questions in the negative by exhibiting a principal ideal which is not an intersection of highest-dimensional primary ideals.

I. Kaplansky (Chicago, Ill.).

Nakayama, Tadasi. Über einfache distributive Systeme unendlicher Ränge. Proc. Imp. Acad. Tokyo **20**, 61–66 (1944). [MF 14872]

The paper contains results on simple rings (not necessarily associative or of finite order), essentially the same as some of those given independently by Jacobson [Trans. Amer. Math. Soc. **57**, 228–245 (1945); these Rev. **6**, 200].

I. Kaplansky (Chicago, Ill.).

Nakayama, Tadasi, und Azumaya, Gorô. Über einfache distributive Systeme unendlicher Ränge. II. Proc. Imp. Acad. Tokyo **20**, 348–352 (1944). [MF 14900]

[Cf. the preceding review.] Theorems on direct products of fields are generalized to the infinite dimensional case. If L is a division ring with center K and M is a maximal subfield, then $L\times_K M$ is a dense ring of linear transformations in a vector space over L. If L, M are fields over K, and at least one of L, M has a separating transcendence basis over K, then $L\times_K M$ is integrally closed in its quotient ring. Necessary and sufficient conditions are found for $L\times_K M$ to have no nilpotent elements or no divisors of zero.

I. Kaplansky (Chicago, Ill.).

Nakano, Noboru. Über die Umkehrbarkeit der Ideale im Integritätsbereiche. Proc. Imp. Acad. Tokyo **19**, 230–234 (1943). [MF 14815]

Let $\mathfrak{R}$ be a commutative ring with unity element, and let $\mathfrak{K}$ be its total quotient ring (consisting of all a/b, b not a zero-divisor). If $\mathfrak{a}$ is an ideal in $\mathfrak{R}$, then $\mathfrak{a}^{-1}$ consists of all $c\epsilon\mathfrak{K}$ such that $c\mathfrak{a}\subseteqq\mathfrak{R}$; $\mathfrak{a}$ is invertible if $\mathfrak{a}\mathfrak{a}^{-1}=\mathfrak{R}$. It is proved that, if all prime ideals are invertible, then $\mathfrak{R}$ has no zero-divisors, every ideal is invertible, and both chain conditions hold modulo every nonzero ideal. [It may be pointed out that without changing the general idea of the paper one can replace most of the last three pages by the following considerations. That $\mathfrak{R}$ is free from zero-divisors follows from the proof of theorem 4. The essential fact is then that (theorem 9) the set $\{\mathfrak{p}_\alpha\}$ of prime ideals containing a given ideal $\mathfrak{a}\neq(0)$ is finite, which may be seen as follows. Let $\mathfrak{a}=\mathfrak{p}_\alpha^{n(\alpha)}\mathfrak{a}_\alpha$, where $n(\alpha)>0$, $\mathfrak{a}_\alpha\not\equiv 0\ (\mathfrak{p}_\alpha)$ (possible by theorem 5). The union of all $\mathfrak{a}_\alpha$ is not contained in any prime ideal, hence is $\mathfrak{R}$; hence $\mathfrak{R}$ must be the union of a finite number $\mathfrak{a}_{\alpha_1}, \cdots, \mathfrak{a}_{\alpha_m}$. If there were a $\beta\neq\alpha_1, \cdots, \alpha_m$, then $\mathfrak{p}_\beta$ would contain $\mathfrak{a}_{\alpha_1}, \cdots, \mathfrak{a}_{\alpha_m}$, a contradiction.] *I. S. Cohen.*

Levitzki, Jakob. On three problems concerning nil-rings. Bull. Amer. Math. Soc. 51, 913–919 (1945). [MF 14460]

The author shows that the first of the following statements implies the other two. (1) A nil-ring generated by a finite number of elements is nilpotent. (2) Every ring has a radical in the sense of Köthe [Math. Z. 32, 161–186 (1930)]. (3) In a ring with distinct upper and lower radicals there exists an ideal between them that is not a radical ideal as defined by Baer [Amer. J. Math. 65, 537–568 (1943); these Rev. 5, 88]. *I. Kaplansky* (Chicago, Ill.).

Shoda, Kenjiro. Bemerkungen über Ringelemente. Proc. Imp. Acad. Tokyo 17, 328–329 (1941). [MF 14715]

Consider a ring R with identity 1, satisfying the double chain condition for right ideals. The author proves the following theorem which contains several criteria as special cases: if J is a right ideal in R, and if a is an element in J such that $aJ=J$, then there exists a left identity element u in the subring J of R such that the element a is a unit of the subring uJu of R. Such elements a in R are called semi-units of R. This is used to prove the following theorem: if the radical N of R is not 0, then R/N is the direct sum of two fields (commutative or not) if, and only if, every element in R is either nilpotent or a semi-unit and R possesses semi-units which are not units. *R. Baer.*

Otobe, Yosikazu. On quasi-evaluations of compact rings. Proc. Imp. Acad. Tokyo 20, 278–282 (1944). [MF 14894]

The author proves that, if R is a compact associative ring which does not contain any total zero divisor other than 0, then we can find a system A of pseudo-evaluations $|x|_\alpha$ of R such that $x=0$ if and only if $|x|_\alpha=0$ for all α in A. A pseudo-evaluation $|x|$ is a real-valued function such that $0\leqq|x|=|-x|\leqq 1$, $|x+y|\leqq\max(|x|, |y|)$, $|xy|\leqq\min(|x|, |y|)$. [The proof is unnecessarily complicated, and incorrect at places.] *C. Chevalley.*

Otobe, Yosikazu. Note on locally compact simple rings. Proc. Imp. Acad. Tokyo 20, 283 (1944). [MF 14895]

Jacobson and Taussky have proved that a locally compact separable simple ring is either connected or totally disconnected and that, in the first case, it is an algebra of finite rank over the field of real numbers [Proc. Nat. Acad. Sci. U. S. A. 21, 106–108 (1935)]. The author proves that the assumption of separability can be removed.

C. Chevalley (Princeton, N. J.).

Otobe, Yosikazu. On locally compact fields. Jap. J. Math. 19, 189–202 (1945). [MF 15002]

Let F be a field (not necessarily commutative) on which there is given a topology which satisfies the following conditions: (a) multiplication and addition are continuous mappings of $F\times F$ into F; (b) the mapping $x\rightarrow x^{-1}$ is continuous at every $x\neq 0$ in F; (c) the first countability axiom holds in F; (d) F is locally compact; (e) F is totally disconnected.

Under these assumptions, Jacobson has proved that F is either a division algebra of finite rank over the field of p-adic numbers or a field of power series in one variable x over a finite field K in which the multiplication is modified by the convention that $xax^{-1}=a^s$, where $a\epsilon K$ and s is an automorphism of K [Amer. J. Math. 58, 433–439 (1936)].

The author proves that the assumptions (b) and (c) can be removed without extending the domain of the fields under consideration. The methods are similar to those used by Jacobson. A compact open subring R of F is constructed and the set P of those $b\epsilon R$ for which $bR=R$ is introduced. It is shown that P is also the set of those $b\epsilon R$ for which $\lim_{n\to\infty} b^n=0$, that P is a two-sided ideal and that the powers of P form a fundamental system of neighborhoods of 0 (which proves that (c) holds). Then it is shown that (b) holds by proving that $(1+P^n)^{-1}\subset 1+P^n$. *C. Chevalley.*

Iyanaga, Shôkichi, und Abe, Makoto. Über das Helmholtzsche Raumproblem. Proc. Imp. Acad. Tokyo 19, 174–180 (1943). [MF 14809]

Iyanaga, Shôkichi, und Abe, Makoto. Über das Helmholtzsche Raumproblem. II. Proc. Imp. Acad. Tokyo 19, 540–543 (1943). [MF 14853]

Suppose that K is an algebraically ordered (commutative) field and that R^n is the n-dimensional affine space over K. If A is an i-dimensional linear subspace of R^n, then A is divided into two i-dimensional half-spaces by every $(i-1)$-dimensional subspace of A. If A_i is, for $0<i\leqq n$, an i-dimensional half-space of R^n, A_0 is a point and A_i is, for $0\leqq i<n$, on the boundary of A_{i+1}, then the A_i are said to form a chain of R^n. Suppose now that G is a group of linear transformations of R^n, meeting the following three requirements. (I) If the linear subspace S of R^n is mapped upon itself by G, and if the transformation g in G leaves invariant every member of a certain chain of S, then g leaves invariant every point in S. (II) If the linear subspace S of R^n is mapped upon itself by G, then there exists to every pair of chains of S a transformation in G which maps the one chain upon the other. (III) If s and t are half-lines with the same endpoint, then there exists a transformation in G interchanging s and t. Then the authors prove (a) if there exists a group G of linear transformations of R^n which meets requirements (I), (II) and (III), then K is Pythagorean in the sense that there exists to every a and b in K an element x in K such that $a^2+b^2=x^2$. (b) To every group G meeting the requirements (I), (II), (III) there exists a distance definition in R^n such that G is the group of all the distance preserving linear transformations of R^n. If, in particular, K happens to be the field of all real numbers, then the authors are able to show that (b) contains as a special case a theorem of H. Weyl [Mathematische Analyse des Raumproblems, Springer, Berlin, 1923, p. 31, theorem T_n].

R. Baer (Urbana, Ill.).

Cohen, I. S. Note on a note of H. F. Tuan. Bull. Amer. Math. Soc. 52, 175–177 (1946). [MF 15458]

The author gives a simplified proof of a theorem of Tuan on replicas of nilpotent matrices [same Bull. 51, 305–312 (1945); these Rev. 7, 3]. *I. Kaplansky* (Chicago, Ill.).

Matsushima, Yozô. Bemerkungen über die Liesche Ringe mit Primzahlcharakteristik. Proc. Imp. Acad. Tokyo 19, 275–277 (1943). [MF 14824]

Let L be a Lie algebra which is the direct sum of simple (non-Abelian) subalgebras and let $D(L)$ be the algebra of derivations of L. It is proved that any subalgebra R of $D(L)$ which contains the algebra of inner derivations has the property that its ideals are all semi-simple. Conversely, this property characterizes the algebras which can be obtained in the way indicated above (if R is given, L is taken to be the sum of the minimal ideals of R). Making use of results and of an example constructed by Jacobson, the author obtains a counter-example to a conjecture of Zassenhaus to the effect that a semi-simple Lie algebra which coincides with its own derived algebra is representable as a sum of simple subalgebras. *C. Chevalley.*

Nakayama, Tadasi, und Matsushima, Yozô. Über die multiplikative Gruppe einer p-adischen Divisionsalgebra. Proc. Imp. Acad. Tokyo 19, 622–628 (1943). [MF 14860]

Let D be a division algebra over the p-adic number field. The center of D is then a $\mathfrak{p}$-adic number field K and D has an absolute-irreducible representation by matrices over K. The reduced norm of each element of D is the determinant of the corresponding matrix. Clearly, any commutator $aba^{-1}b^{-1}$ in D has reduced norm 1. The authors prove, conversely, that every element a of D with reduced norm $N(a)=1$ is necessarily a product of commutators (actually, a product of at most three commutators). The element a is a unit of D, and the proof is obtained by a convergence argument on units congruent to 1 modulo successively higher powers of a prime element. Using results of M. Abe [Sijo-Sugaku-Danwakai **240** (1942)], the same theorem is obtained for normal simple algebras over a $\mathfrak{p}$-adic field. The authors also obtain a representation of all units of D as a product of p-adic powers of a finite number of units of D. This result was obtained by O. F. G. Schilling [Amer. J. Math. 61, 883–896 (1939); these Rev. 1, 38] in a form not explicitly using the center of D. *S. MacLane.*

***Brauer, Richard. On hypercomplex arithmetic and a theorem of Speiser.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 233–245, Füssli, Zürich, 1945.

Let A be a Frobenius algebra of rank n over an algebraic number field K admitting F as an absolutely irreducible representation of A. Let J be an "order" of A, and let $J_0=\mathfrak{o}_0J$, where $\mathfrak{o}_0$ is the ring of $\mathfrak{p}$-integers of K, $\mathfrak{p}$ being a prime ideal of the ring of integers of K. The representation is said to be "without defect" if a certain set of conditions holds; it is shown that, for a given J, F is without defect except for a finite number of prime ideals $\mathfrak{p}$ of $\mathfrak{o}$, and that the corresponding modular representation $\bar{F}$ is absolutely irreducible in $\bar{K}=\mathfrak{o}/\mathfrak{p}$. If F is without defect for $\mathfrak{p}$, and if F_λ is a different irreducible representation of A, then $\bar{F}_\lambda$ does not contain $\bar{F}$ as a modular irreducible constituent. We may assume that the corresponding indecomposable part U of the regular representation represents J by $\mathfrak{p}$-integral matrices; if a modular representation T of J splits into $\bar{U}$ and another representation V, then T splits completely into $\bar{U}$ and V. This generalizes a theorem of Speiser [Theorie der Gruppen von Endlicher Ordnung, Springer, Berlin, 1923]. Let F be a representation of A without defect for the prime ideal $\mathfrak{p}$ of K and the order J of A, and write F so that it represents elements of F by matrices with $\mathfrak{p}$-integral coefficients. The elements α of A with $F(\alpha)=0$ form a maximal invariant subalgebra B of A, and the elements ξ of J with $F(\xi)\equiv 0 \pmod{\mathfrak{p}}$ form a prime ideal divisor $\mathfrak{P}$ of $\mathfrak{p}$ in J. Then $\mathfrak{P}=(B\cap J, \mathfrak{p}J)$. *C. C. MacDuffee* (Madison, Wis.).

Jacobson, N. Structure theory for algebraic algebras of bounded degree. Ann. of Math. (2) **46**, 695–707 (1945). [MF 14129]

An algebra $\mathfrak{A}$ over a field ϕ is said to be algebraic if every element of $\mathfrak{A}$ satisfies an algebraic equation with coefficients in ϕ. If there exists an integer $N>0$ such that every element satisfies an equation of degree at most N, $\mathfrak{A}$ is said to be of bounded degree. If we assume only that $x^N=0$ for every nilpotent x, $\mathfrak{A}$ is said to be of bounded index. The algebra $\mathfrak{A}$ is called primitive if it contains a right ideal $\mathfrak{J}$ such that $\mathfrak{A}:\mathfrak{J}=\{0\}$ (this is a generalization of the notion of simple algebra). It is proved that any primitive algebraic algebra of bounded index is isomorphic with the algebra of matrices of a certain degree with coefficients in an algebraic division algebra. From this, one can obtain information on algebraic algebras of bounded indices which are semi-simple in the sense of Jacobson [Amer. J. Math. **67**, 300–320 (1945); these Rev. **7**, 2].

An algebraic division algebra of bounded degree is shown to be of finite dimension over its center, and therefore also over the basic field if the center is separable over the basic field. As an application it is proved that every algebraic algebra over a finite field ϕ is commutative; the same is true of any algebraic algebra over ϕ which has no nilpotent element except 0. Generalizing the theorem that any Boolean ring is commutative, the author proves that, if $\mathfrak{R}$ is a ring in which every element satisfies an equation of the form $a^{n(a)}=a$, $n(a)>1$, then $\mathfrak{R}$ is commutative.

If now we consider a semi-simple algebra $\mathfrak{A}$ of bounded degree, then it is shown that, if the basic field ϕ is perfect and infinite, $\mathfrak{A}$ is finite over ϕ. Even if ϕ is finite, $\mathfrak{A}$ is locally finite (that is, every finite subset of $\mathfrak{A}$ generates a subalgebra of finite dimension). *C. Chevalley.*

THEORY OF GROUPS

Jessen, Axel. A method for the study of finite groups. Mat. Tidsskr. B. **1941**, 41–42 (1942). (Danish) [MF 13588]

The author sketches briefly a method for the determination of all subgroups of a finite group, based on a study of the chains of the group. Any two elements $g_1=a$ and $b=g_2$ define a chain $g_3=g_1\cdot g_2=ab$, $g_4=g_2\cdot g_3=bab$, $\cdots$, $g_{r+1}=g_{r-1}\cdot g_r$. It is not clear to the reviewer why the chain should revert cyclically ($g_{k+1}=a$, $g_{k+2}=b$) for some k as stated by the author. Each pair of elements defines a chain and every subgroup consists of a certain number of chains. The chain is called typical for the least subgroup containing it, that is, the subgroup generated by a and b. *O. Ore.*

Brauer, Richard. On the representation of a group of order g in the field of the g-th roots of unity. Amer. J. Math. 67, 461–471 (1945). [MF 13934]

All the representations of a finite group of order g cannot in general be written in terms of rational numbers, some irrationals being necessary. It is of interest to determine the minimum extension of the rational field necessary to obtain all matrix representations. It has long been conjectured that the adjunction of a primitive gth root of unity would be sufficient in all cases. Maschke, Burnside, Schur, Speiser and Hasse have proved theorems directed towards confirmation. In this paper the author gives the first complete proof. His statement of the theorem is as follows: if G is a group of finite order g, then every irreducible representation L of G can be written in the field Ω of the gth roots of unity (that is, L is similar to a representation with coefficients in Ω).

The proof for modular representations follows in a few lines from a previous theorem of the author. The extension to ordinary representations is the main part of the paper. It is accomplished with reference to the author's earlier work on the connection between ordinary and modular representations of a group, by intricate work concerning the theory of ideals. Since, for representations with character-

istic 0, all reducible representations are completely reducible, for these the theorem holds also for reducible representations. An example is given which shows that the theorem fails for modular representations which are reducible but not completely reducible. *D. E. Littlewood* (Swansea).

Gårding, Lars. On a class of linear transformations connected with group representations. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 6, 1–125 (1944). [MF 13599]

If $\alpha=[a_{st}]$ is an orthogonal transformation and $S(\alpha)$ is a basic spin representation, the matrix equation

$$S(\alpha)U_jS^{-1}(\alpha)=\sum a_{kj}U_k$$

has for solution the Dirac matrices. This thesis is an attempt at generalisation in which $S(\alpha)$ can be any matrix representation and the further equations

$$S(\alpha)U_jT^{-1}(\alpha)=\sum a_{kj}U_k,$$
$$S(\alpha)U_{jk}T^{-1}(\alpha)=\sum a_{jj'}a_{kk'}U_{j'k'},$$
$$S(\alpha)W_{jk}S^{-1}(\alpha)=\sum a_{jj'}a_{kk'}W_{j'k'}$$

are considered. The results obtained are not fundamental as are the Dirac matrices and the corresponding spinors, but an efficient technique is developed over a specialized field.

In the case of the last equation it is observed that $W_{jk}=\delta_{jk}I$ is one solution, δ_{jk} being the Kronecker symbol, and that, if other solutions are known, further solutions may be obtained from them by the process of tensor contraction. A detailed discussion is given of the solutions, including the number of linearly independent solutions and the connection between a given set of linearly independent solutions and their transposes. The discussion includes a considerable account of known results in group character theory and the theory of infinitesimal transformations. Generalisation to the full linear groups and the symplectic groups are also discussed.

Finally, application is made to differentiation in n-space under the orthogonal group. Let ϕ be a function of position of simple type $[\lambda]$, and Δ_i a linear differential operator, as, for example, div, curl or grad, such that $\Delta_i\phi=\psi$ is a function of position of simple type $[\mu]$. For each such operator Δ_i there is another such operator Δ_i' such that $\Delta_i'\psi=\phi'$ is of the same type $[\lambda]$ as ϕ. The number of operators Δ_i for given type $[\lambda]$ is discussed and also the number of linear combinations of the second order differential terms $\Delta_i'\Delta_i\phi$ which are equivalent to $\nabla^2\phi$, where ∇^2 is the invariant Laplacean operator. Hence the possible sets of first order differential equations which lead to the second order equations $\nabla^2\phi=0$ and $\nabla^2\phi=k^2\phi$ are discussed. Here some contact is made with quantum theory and nuclear physics.

D. E. Littlewood (Swansea).

Kaloujnine, Léo. Sur les p-groupes de Sylow du groupe symétrique du degré p^m. C. R. Acad. Sci. Paris 221, 222–224 (1945). [MF 14252]

Denote by A_m the set of all vectors $[x_1, \cdots, x_m]$ with components in $GF(p)$, and consider all permutations of A_m of the form

$$[x_1, \cdots, x_m]\rightarrow[x_1+a_0, x_2+a_1(x_1), \cdots, x_m+a_{m-1}(x_1, \cdots, x_{m-1})],$$

where $a_0\epsilon GF(p)$ and $a_k(x_1, \cdots, x_k)$ is any function of the components $x_1, \cdots, x_k$ with values in $GF(p)$. It is shown that these permutations form a transitive p-group which is isomorphic to the Sylow subgroups of the symmetric group of degree p^m. A generalization to permutations of sets of the form $[g_1, \cdots, g_m]$, where $g_1, \cdots, g_m$ belong to the abstract groups $G_1, \cdots, G_m$, respectively, yields a group which the author calls the "complete product" of the groups $G_1, \cdots, G_m$. *S. A. Jennings* (Vancouver, B. C.).

Siegel, Carl Ludwig. Some remarks on discontinuous groups. Ann. of Math. (2) 46, 708–718 (1945). [MF 14130]

Let Ω be either the group of all real transformations of the form $x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_m^2$ into itself, or the real symplectic group of degree p, where $2p=m$. In both cases, if the matrix C belongs to Ω, so does its transpose C'. The first part of the paper deals with the structure of the homogeneous space $\mathfrak{T}$ of all positive symmetric T in Ω and the representation of Ω as a transformation group over $\mathfrak{T}$. It is shown that $\mathfrak{T}$ is homeomorphic to the space of cosets of Ω modulo a certain maximal compact subgroup and that distinct elements C_1 and C_2 of Ω define distinct transformations $C_1'TC_1\rightarrow T$ and $C_2'TC_2\rightarrow T$ of $\mathfrak{T}$ unless $C_1=\pm C_2$. If $\mathfrak{P}$ is the space of all positive symmetric matrices T of degree m, with the metric $\dot{s}^2=\text{trace}\ (\dot{T}T^{-1}\dot{T}T^{-1})$, then $\mathfrak{T}$ is a geodesic manifold of $\mathfrak{P}$ of nonpositive Gaussian curvature, and lower bounds are obtained for the volumes of geodesic spheres in $\mathfrak{T}$. The results are obtained by means of a parametrization of the matrices of $\mathfrak{T}$ in terms of arbitrary matrices of certain types, which is of intrinsic interest. The second part of the paper is devoted to a discussion of the fundamental regions of the discontinuous representations in $\mathfrak{T}$ of discrete subgroups Γ of Ω, along lines introduced previously by the author [same Ann. (2) 44, 674–689 (1943); these Rev. 5, 228] in connection with more general topological groups. The representation of Γ as transformations in $\mathfrak{T}$ is shown to be faithful if $-E$ is not contained in Γ; otherwise a matrix of Γ and its negative define the same transformation. Upper bounds for the geodesic radius of the fundamental regions of discrete subgroups are obtained and for certain types of subgroup the volumes of these regions are shown to have a positive lower bound. A detailed application of these ideas to groups of motions in the non-Euclidean plane yields several interesting results: for example, the area J of the fundamental region is at least $\pi/21$, and if J is finite then Γ may be generated by at most $6+3\pi^{-1}J$ elements. *S. A. Jennings* (Vancouver, B. C.).

Iwasawa, Kenkiti. Einige Sätze über freie Gruppen. Proc. Imp. Acad. Tokyo 19, 272–274 (1943). [MF 14823]

Theorem: if $A\neq1$ is an element of the free group F, there exists a normal subgroup of finite index in F which does not contain A. The proof depends on a similar theorem for nilpotent groups with a finite number of generators, and on the theorem of Magnus [J. Reine Angew. Math. 177, 105–115 (1937)] that all groups Z_n of the descending central series of a free group have intersection 1. The author draws some immediate consequences about almost periodic functions in the sense of von Neumann [Trans. Amer. Math. Soc. 36, 445–492 (1934)]. *S. MacLane.*

Takahasi, Mutuo. Bemerkungen über den Untergruppensatz in freien Produkte. Proc. Imp. Acad. Tokyo 20, 589–594 (1944). [MF 14930]

Suppose that U is a subgroup of the free product $G=\prod^*G(v)$. Denote by $r(v, w)$ for $0\leqq w<d(v)$ a set of representatives of the decomposition of G with respect to the modulus $(U, G(v))$ so that $G=UG(v)+Ur(v, 0)G(v)+\cdots+Ur(v, w)G(v)+\cdots$. Then U may be decomposed into a free product as follows:

$$U=F^*\prod_v{}^*\prod_{0\leqq w<d(v)}{}^*\{U\cap r(v, w)G(v)r(v, w)^{-1}\},$$

where F is a free group. If, in particular, U is of finite index j in G, and if n is the (finite or infinite) number of factors $G(v)\neq 1$, then the rank of the free group is $1+jn-j=\sum_v d(v)$. This theorem constitutes an improvement of the subgroup theorems of Kurosch [Math. Ann. **109**, 647–660 (1934)] and Baer-Levi [Compositio Math. **3**, 391–398 (1936)]. Its proof is effected by an adaptation of the combinatorial methods used in the proof of Schreier's theorem [see, in particular, K. Reidemeister, Einführung in die Kombinatorische Topologie, Vieweg, Braunschweig, 1932; F. Levi, Math. Z. **32**, 315–318 (1930); **37**, 90–97 (1933)].

R. Baer (Urbana, Ill.).

Nakayama, Tadasi. Note on lattice-ordered groups. Proc. Imp. Acad. Tokyo **18**, 1–4 (1942). [MF 14734]

It is shown that every l-group G is a distributive lattice, and that every l-group satisfying the chain condition is a direct product of specimens of the additive group of integers. [Cf. theorems 14, 37 of the reviewer's paper in Ann. of Math. (2) **43**, 298–331 (1942); these Rev. **4**, 3.] It is further proved that, if G has an Archimedean unit 1, then an element x is congruent to 0 modulo every maximal l-ideal if and only if $n|x|<1$ for all n. *G. Birkhoff.*

Nakayama, Tadasi. Note on free topological groups. Proc. Imp. Acad. Tokyo **19**, 471–475 (1943). [MF 14845]

This note, inspired by a concept due to A. Markoff, appears to overlap some of the more recent work of that writer on free topological groups [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 3–64 (1945); these Rev. **7**, 7]. The author shows that to each element, not the identity, of a free topological group F (generated by a completely regular space R) there may be associated a continuous unitary representation distinguishing that element from the identity. The totality of these functions on F topologizes F to a totally bounded group F^* (also generated by R) and R is closed both in F and in F^*. The notion of a free topological group is extended to that of a uniform free topological group, generated by a uniform space R satisfying Hausdorff's separation axiom.

L. Zippin (Flushing, N. Y.).

Kakutani, Shizuo. Free topological groups and infinite direct product topological groups. Proc. Imp. Acad. Tokyo **20**, 595–598 (1944). [MF 14931]

This note contains a new proof of Markoff's theorem that there exists a free topological group generated by a completely regular space, a proof of a theorem of Nakayama [see the preceding review] and a generalization of the concept of free topological group. *L. Zippin.*

Kodaira, Kunihiko, and Kakutani, Shizuo. Normed ring of a locally compact Abelian group. Proc. Imp. Acad. Tokyo **19**, 360–365 (1943). [MF 14832]

Let G be a locally compact (not necessarily separable) Abelian group and let $L^2(G)$ be the Hilbert space of all complex-valued functions defined and of integrable square on G with respect to a Haar measure (fixed normalization) on G. Denote by $\mathfrak{B}(G)$ the normed ring of all bounded linear operators on $L^2(G)$ and let $\mathfrak{R}(G)$ denote the smallest closed subring of $\mathfrak{B}(G)$ which contains all unitary operators of the form $U_a x=x_a$, where $x(g)\in L^2(G)$ and $x_a(g)=x(g-a)$. The authors show that there exists a one-to-one correspondence between the class $\mathfrak{M}(G)$ of all maximal ideals in $\mathfrak{R}(G)$ and the class $\mathfrak{X}(G)$ of all algebraic (that is, not necessarily continuous) characters of G. If the usual weak topology is used in $\mathfrak{M}(G)$ and if $\mathfrak{X}(G)$ is considered as the compact character group of a discrete group algebraically isomorphic to G, then the correspondence is a homeomorphism. Also $\mathfrak{R}(G)$ is isometrically isomorphic with the normed ring $C(\mathfrak{M}(G))$ of all complex-valued continuous functions on $\mathfrak{M}(G)$. These facts show that $\mathfrak{R}(G)$ is uniquely determined (up to an isomorphism) by the algebraic structure of G.

C. E. Rickart (New Haven, Conn.).

Yosida, Kôsaku. On the duality theorem of non-commutative compact groups. Proc. Imp. Acad. Tokyo **19**, 181–183 (1943). [MF 14810]

Tannaka [Tôhoku Math. J. **45**, 1–12 (1938)] has proved a duality theorem for compact non-Abelian groups. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 9–12 (1941); these Rev. **2**, 316] and Bochner [Ann. of Math. (2) **43**, 56–58 (1942); these Rev. **3**, 208] have also proved this theorem. The present paper gives still another proof. Let G be a compact Hausdorff group and let R be the ring of Fourier polynomials in the elements of a set of mutually inequivalent, continuous, unitary, irreducible representations of G. Let T be the set of linear homomorphisms t taking R into the field of complex numbers with $t\cdot e=1$ and $t\cdot\bar{x}=\overline{t\cdot x}$. Then T is isomorphic to G. *D. Montgomery.*

Yosida, Kôsaku, and Iwamura, Turane. Equivalence on two topologies of Abelian groups. Proc. Imp. Acad. Tokyo **20**, 451–453 (1944). [MF 14912]

Proof that two topologies on the character group G^* of a locally compact Abelian group G are the same, one topology being the usual compact-open topology, the other being the topology induced on G^* by the weak topology on the linear functions on $L_1(G)$ (every $x^*\in G^*$ is identifiable in an obvious way with a linear functional on $L_1(G)$). This is proved only for separable groups, though it is proved even without separability that G^* is a topological group in the second topology. *W. Ambrose* (Ann Arbor, Mich.).

Iwasawa, Kenkichi. On group rings of topological groups. Proc. Imp. Acad. Tokyo **20**, 67–70 (1944). [MF 14873]

Extensions of some theorems of I. E. Segal [Proc. Nat. Acad. Sci. U. S. A. **27**, 348–352 (1941); these Rev. **3**, 36], mainly concerned with the correspondence between representations of a group G and representations of its group ring, in the case where G is locally compact. The author considers the ring $L^{(1,p)}(G)$ of all functions x in $L_1(G)\cap L_p(G)$ with $\|x\|=\max(\|x\|_1, \|x\|_p)$ and convolution for ring multiplication. No unit is adjoined to the ring. For finite dimensional (matrix) representations the following theorem is proved. There exists a one-to-one correspondence between continuous representations $x\rightarrow T(x)$ of this ring and bounded continuous representations $g\rightarrow D(g)$ of G, corresponding representations being related by the formula $T(x)=\int x(g)D(g)dg$, and this correspondence preserves equivalence of representations. Some related theorems are given concerning reducibility, measurability and continuity of representations, and a similar theorem is given for infinite dimensional (operator) representations. *W. Ambrose.*

Kawada, Yukiyosi. Über den Mittelwert der messbaren fastperiodischen Funktionen auf einer Gruppe. Proc. Imp. Acad. Tokyo **19**, 264–266 (1943). [MF 14820]

J. von Neumann has shown how to define the mean value of an almost periodic function on an arbitrary group [Trans. Amer. Math. Soc. **36**, 445–492 (1934)]. This paper proves that for certain topological groups the mean $M[f(x)]$, de-

fined by von Neumann, is given by a formula which is an extension of the case where the function is defined for real numbers. The exact statement follows. Let G be an Abelian locally compact connected group and let m be the Haar measure of G. Let D be an arbitrary open totally bounded set in G, and let E_n be the subset generated by adding all sets of n elements in D. Then, for any measurable almost periodic function $f(x)$ on G,

$$M[f(x)] = \lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} f(x)m(dx).$$

D. Montgomery (Princeton, N. J.).

Godement, Roger. Sur la presque-périodicité des fonctions spectrales. C. R. Acad. Sci. Paris **221**, 686–687 (1945). [MF 15158]

The following result is announced: if f and f' are arbitrary functions in the class L on a locally compact topological group G, then the function $f\times f'(x) = M(f_x f')$ is almost periodic on G. *M. H. Stone* (Cambridge, Mass.).

Abe, Makoto. Sur la métrique riemannienne et l'élément de volume dans les espaces de groupes de Lie. Proc. Imp. Acad. Tokyo **19**, 629–634 (1943). [MF 14861]

It is shown that a necessary and sufficient condition for a Lie group G to admit a two-sided invariant Riemannian metric is that G is the product of a compact and an Abelian group. A necessary and sufficient condition for G to admit a two-sided invariant volume element is that the operations of the linear adjoint group of G are of determinant 1. Then the affine connection without torsion defined by Cartan on G is investigated [cf. É. Cartan, J. Math. Pures Appl. (9) **6**, 1–119 (1927)]; in particular, the question is raised under which condition this connection can be defined by a Riemannian metric. A necessary and sufficient condition for this is found to be that G is locally the product of a semisimple compact group and a nilpotent group of class not greater than 2. However, the author seems to consider it evident that a sufficient condition for an affine connection without torsion to be definable by a Riemannian metric is that its group of holonomy should leave a positive definite quadratic form invariant, a statement which, in the opinion of the reviewer, is far from obvious. *C. Chevalley.*

Shizuma, Ryoji. Über die Untergruppen geschlossener Liescher Gruppen. Proc. Imp. Acad. Tokyo **20**, 670–673 (1944). [MF 14942]

The principal result is the theorem that, if G is a closed Lie group of rank r and G_i, $i=0, 1, \cdots, m$, are closed subgroups with $G_0=G$ and G_m the identity, such that G_{i+1} is a proper subgroup of G_i and G_{i+1} is not homologous to zero (in the combinatorial sense) in G_i, then $m\leqq r$. A second theorem is that the only connected subgroups of the rotation-group of the two-sphere are its "circle" subgroups. However, it seems to the reviewer that the (very easy) proof requires that the subgroups in question are closed and that the author may not be making allowance for this in one of his footnote remarks. *L. Zippin* (Flushing, N. Y.).

Bochner, Salomon, and Montgomery, Deane. Groups of differentiable and real or complex analytic transformations. Ann. of Math. (2) **46**, 685–694 (1945). [MF 14128]

(1) Soit D une des classes suivantes: la classe C^k ($k>1$), la classe réelle analytique, la classe complexe analytique. Soit G un groupe locale de Lie à paramètres réels analytiques (dans le dernier cas: à paramètres complexes analytiques) agissant dans une variété D. Si G est de classe D pour chacque valeur des paramètres, il est aussi de classe D simultanément par rapport aux paramètres et aux coordonnées. Le théorème est démontré avec la méthode de l'intégration dans les sous-groupes à un paramètre.

(2) Si G est un groupe de Lie compact et à paramètres complexes analytiques, G est abélien. Ce théorème appartient au même ordre d'idées que le théorème de Liouville sur les fonctions analytiques complexes. Généralisation : on ne suppose que la compacticité du groupe adjoint, et on admet des fonctions harmoniques comme coefficients du groupe adjoint. *H. Freudenthal* (Amsterdam).

Dieudonné, Jean. Sur la complétion des groupes topologiques. C. R. Acad. Sci. Paris **218**, 774–776 (1944). [MF 15326]

Bourbaki [Topologie générale, Actual. Sci. Ind., no. 916, Hermann, Paris, 1942, p. 28; these Rev. **5**, 102] has asked whether a topological group is always isomorphic to a subgroup of a complete group. It is shown here by means of two simple examples that this is not always the case.

D. Montgomery (Princeton, N. J.).

Vilenkin, N. J. A contribution to the theory of direct decompositions of topological groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 611–613 (1945). [MF 14412]

Each topological (additive) Abelian group G_i of a given sequence is assumed to be locally compact, to satisfy the second countability axiom, and to have the component of its identity element compact. In G_i an arbitrary open compact subgroup H_i is distinguished. The direct sum $G=\sum(G_i, H_i)$ with the subgroups H_i distinguished is then defined as the group of all sequences $\{x_i\}$, where $x_i\epsilon G_i$ for all i and $x_i\epsilon H_i$ for almost all i, topologized by the assignment to the identity of the neighborhood system consisting of the subsets $U(U_1, \cdots, U_r)$ characterized by the conditions $x_i\epsilon U_i$, $i=1, \cdots, r$, and $x_i\epsilon H_i$, $i>r$, where U_i is a neighborhood of the identity in G_i. As a topological group, G is stated to share the properties assumed for G_i; in it the subgroup H characterized by the conditions $x_i\epsilon H$, $i=1, 2, \cdots$, is found to be an open compact subgroup such that $H_i = H\cap G_i$. The conditions are given under which a group G with distinguished subgroup H has a representation as a direct sum of the above type in terms of subgroups G_i with distinguished subgroups $H_i=H\cap G_i$. The principal results stated concern the direct-sum representations of zero-dimensional groups G in which every element generates a compact subgroup. Every such group G, with a distinguished subgroup H, is the direct sum of groups (termed topologically primary) for each of which there is a unique prime p with the property that $p^n x\rightarrow 0$ for all x. The topologically primary groups are asserted to have, under appropriate sets of characteristic conditions, given in full, representations as direct sums of (a) cyclic groups; (b) additive groups of p-adic integers; (c) groups of type (p^∞) [see H. Prüfer, Math. Z. **17**, 35–61 (1923), in particular, p. 41]; (d) additive groups of rational p-adic numbers. *M. H. Stone.*

Toyoda, Kôshichi. On axioms of linear functions. Proc. Imp. Acad. Tokyo **17**, 221–227 (1941). [MF 14705]

Let G be a set of elements $a, b, c, \cdots$ which satisfies the following axioms: (1) the product ab of any two elements is an element of G; (2) if a, b, c, d are any four elements of G, then $(ab)(cd) = (ac)(bd)$; (3) there exist unique elements x

and y of G such that $xa=b$, $ay=b$, for any two elements a, b of G. It is shown that the elements of G form an Abelian group under the new operation $+$, defined by $as+rb=ab$, where r and s are two fixed elements of G. Moreover, the original multiplication can be expressed in terms of the new operation by $xy=Ax+By+c$, where A and B are commutative automorphisms of G (with respect to both operations) and c is a fixed element of G. [This result has also been proved in part by the reviewer [Trans. Amer. Math. Soc. **49**, 392–409 (1941); these Rev. **2**, 218] and in toto by R. H. Bruck [Trans. Amer. Math. Soc. **55**, 19–52 (1944), p. 46, theorem 12; these Rev. **5**, 229].]

Two generalizations are given. First, axiom (3) is replaced by a weaker axiom which postulates the existence of a "unit element" 0, such that $0\cdot 0=0$ and the equations $x0=a$, $0y=a$ have at least one solution for x and y, a being any element of G. A new operation is then defined under which the elements of G form a commutative semi-group. Finally, the author considers a system G in which is defined an $n$ary operation which associates with each set of n elements $a_1, \cdots, a_n$ of G an element $a_0=(a_1, \cdots, a_n)$ of G. This operation satisfies the associative law

$$[(a_1, \cdots, a_n),\ (b_1, \cdots, b_n),\ \cdots,\ (d_1, \cdots, d_n)] \\ =[(a_1, \cdots, d_1),\ (a_2, \cdots, d_2),\ \cdots,\ (a_n, \cdots, d_n)]$$

and the quotient axiom: each of the equations $(x,b,a,\cdots,a)=c$ and $(b,x,a,\cdots,a)=c$ is uniquely soluble for x. A binary operation $+$, under which the elements of G form an Abelian group, is then defined in terms of this $n$ary operation. Moreover, the original $n$ary operation can be expressed in terms of the operation $+$ and certain automorphisms and homomorphisms of G. *D. C. Murdoch.*

NUMBER THEORY

Cotlar, Mischa. A generalization of the factorials. Math. Notae **5**, 89–107 (1945). (Spanish) [MF 14101]

The author defines $f_{n,r}$ as the number of permutations of $n+r$ letters $a_1, \cdots, a_r, b_1, \cdots, b_n$ such that no one of the letters b_i is left fixed. For the case $r=0$ this was studied by Euler and is the number of ways n rooks can be placed on an $n\times n$ chess board so that no one can take any other one and so that none lies on a given diagonal of the board. Many relations between these numbers are derived. The generating function $F_n(x)=\sum f_{n,r}x^r/r!$ is studied and it is shown that $F_n(x)=P_n(x)(1-x)^{-n-1}$, where $P_n(x)$ is a polynomial with the property $dP_n/dx=nP_{n-1}$. Various generalizations are suggested. *H. W. Brinkmann.*

Levi, Beppo. On a Diophantine problem. Math. Notae **5**, 108–119 (1945). (Spanish) [MF 14102]

The paper concerns a problem proposed in the same journal: find integral sides of a right-angled triangle which again form a right-angled triangle when a digit is prefixed. The two answers submitted claimed that 5, 12, 13; 15, 112, 113 is the only solution. The author shows that, on the contrary, there are an infinite number of families of solutions. For example, one such family is $6\cdot 10^{n+2}$, $1125\cdot 10^{2n+1}\pm 8$ for $n=0, 1, \cdots$, with 2 the digit to be prefixed. The method is an elementary case by case reduction. *I. Kaplansky.*

Santaló, L. A. Addendum to the note "On a Diophantine problem." Math. Notae **5**, 162–171 (1945). (Spanish) [MF 14579]

Continuing Levi's analysis [cf. the preceding review], the author shows that the prefixed digit a must satisfy $1\leqq a\leqq 7$, in each of which cases there are an infinite number of solutions. *I. Kaplansky* (Chicago, Ill.).

Wachs, Sylvain. Sur une extension du théorème de Gauss à un système de nombres premiers dans leur ensemble. Bull. Sci. Math. (2) **69**, 46–52 (1945). [MF 14427]

It is proved that, if the integral elements of the first row of a determinant have greatest common divisor 1, then the integral elements of the remaining rows can be chosen so that the value of the determinant is 1. *I. Niven.*

Erdös, Paul. On the coefficients of the cyclotomic polynomial. Bull. Amer. Math. Soc. **52**, 179–184 (1946). [MF 15460]

Denote by A_n the largest absolute value of the coefficients of the cyclotomic polynomial $F_n(x)$, whose roots are the $\phi(n)$ primitive nth roots of unity. E. Lehmer [Bull. Amer. Math. Soc. **42**, 389–392 (1936)] gave Schur's proof that $\limsup A_n=\infty$ and proved that $A_n>cn^{\frac{1}{3}}$ for infinitely many n of the form pqr, with p, q and r distinct primes. The author proves that $\max|F_n(x)|$ for $|x|=1$ is greater than $\exp\{c_2(\log n)^{4/3}\}$ for infinitely many $n=2\cdot 3\cdot 5\cdot\cdots\cdot p_k$, the product of the first k primes. This implies $A_n>\exp\{c_1(\log n)^{4/3}\}$ and hence that $A_n>n^k$, for every k and infinitely many n. The proofs employ lemmas based on interpolation formulas for trigonometric polynomials, the prime number theorem and asymptotic expressions for $\phi(n)$. The author also proves a theorem which suggests that a much stronger result may be true. *R. Hull* (Lincoln, Neb.).

Erdös, P. On a problem of Sidon in additive number theory and on some related problems. Addendum. J. London Math. Soc. **19**, 208 (1944). [MF 14567]

The author describes a relationship between results of J. Singer [Trans. Amer. Math. Soc. **43**, 377–385 (1938)] and results of Erdös and Turán [same J. **16**, 212–215 (1941); these Rev. **3**, 270]. *B. W. Jones* (Ithaca, N. Y.).

Chowla, Inder. On Waring's problem (mod p). Proc. Nat. Acad. Sci. India. Sect. A. **13**, 195–220 (1943). [MF 15178]

Let p denote a prime and k a positive integer. The paper deals mainly with the function $\gamma(k, p)$ defined as the least s such that the congruence $\sum_{m=1}^{s} x_m^{k}\equiv n \pmod p$ is solvable for every n. The main result is as follows. Let $p\equiv 1 \pmod k$, $p>3k$, $C=(103-3\sqrt{641})/220$, and $\epsilon>0$. Then $\gamma(k, p)=O(k^{1-C+\epsilon})$. The author applies this to the function $\Gamma(k)$ introduced by Hardy and Littlewood in connection with Waring's problem [Proc. London Math. Soc. (2) **28**, 518–542 (1928)], and obtains a result of the form $\Gamma(k)=\max\{f(k), g(k)\}$, where $f(k)=O(k^{1-C+\epsilon})$ for every positive ϵ, and $g(k)$ is an elementary arithmetical function. This is an improvement on Hardy and Littlewood's results. *T. Estermann* (London).

Drach, Jules. Sur quelques points de théorie des nombres et sur la théorie générale des courbes algébriques. C. R. Acad. Sci. Paris **221**, 729–732 (1945). [MF 15161]

The author gives, by the method of descent, proofs of well-known theorems on the numbers represented by the sums of two, three and four squares and by the sum of

three triangular numbers. He also gives a means of classification of algebraic curves with certain specified types of singularities. There seems to be little connection between the two parts of the paper. *B. W. Jones* (Ithaca, N. Y.).

Mian, Abdul Majid, and Chowla, S. On the B_2 sequences of Sidon. Proc. Nat. Acad. Sci. India. Sect. A. 14, 3–4 (1944). [MF 14293]

The authors consider the sequence $a_1=1$, $a_2=2$, $a_3, \cdots$, where a_{m+1} is defined as the smallest integer such that $a_{m+1}-a_r \neq a_s - a_t$, for all r, s, t not exceeding n. They conjecture that $a_n^{\frac{1}{3}} \leqq n$. *P. Erdös.*

Chowla, S. Solution of a problem of Erdös and Turan in additive-number theory. Proc. Nat. Acad. Sci. India. Sect. A. 14, 1–2 (1944). [MF 14321]

Chowla, S. Solution of a problem of Erdös and Turan in additive-number-theory. Proc. Lahore Philos. Soc. 6, 13–14 (1944). [MF 14320]

Denote by $M(n)$ the maximum number of integers not greater than n such that the sums of any two are different. By using a result of Singer [Trans. Amer. Math. Soc. 43, 377–385 (1938)], the author proves that lim $(M(n)/\sqrt{n})=1$. *P. Erdös* (Stanford University, Calif.).

Chowla, S. Contributions to the theory of the construction of balanced incomplete block designs used in the statistical tables of Fisher and Yates. Proc. Lahore Philos. Soc. 6, no. 2, 10–12 (1944). [MF 14326]

Chowla, S. Contributions to the theory of the construction of balanced incomplete block designs. Proc. Lahore Philos. Soc. 6, no. 2, 17–23 (1944). [MF 14330]

The author states various interesting results on difference sets, found partly by himself and partly by others. *H. B. Mann* (Columbus, Ohio).

Chowla, S. A property of biquadratic residues. Proc. Nat. Acad. Sci. India. Sect. A. 14, 45–46 (1944). [MF 14324]

A set of numbers $d_1, \cdots, d_k$ is said to form a difference set (mod p) if the congruence $d_i - d_j \equiv a$ (mod p) has the same number of solutions for every a prime to p. The author proves that the biquadratic remainders mod p form a difference set if $(p-1)/4$ is an odd square. *H. B. Mann.*

Chowla, S. A formula similar to Jacobsthal's for the explicit value of x in $p=x^2+y^2$ where p is a prime of the form $4k+1$. Proc. Lahore Philos. Soc. 7, 2 pp. (1945). [MF 15091]

For any prime p of the form $3k+1$, the Diophantine equation $4p=x^2+27y^2$ has a unique solution in x and y except for signs. Choose the sign of x so that $x\equiv 1$ (mod 3). It is proved that

$$x=1+\sum_{m=1}^{p}\left(\frac{m}{p}\right)\left(\frac{m^3-4}{p}\right),$$

where $(\frac{m}{p})$ is Legendre's symbol. [Presumably the author intended the summation to extend only to $m=p-1$.] *I. Niven* (West Lafayette, Ind.).

Chowla, S. The cubic character of 2 (mod p). Proc. Lahore Philos. Soc. 7, 1 p. (1945). [MF 15092]

On the basis of a previous result [see the preceding review] a new proof is given of the theorem that $u^3\equiv 2$ (mod p) is solvable if and only if x is even. *I. Niven.*

Chowla, S. The cubic character of 2 (mod p). Proc. Lahore Philos. Soc. 6, no. 2, 12 (1944). [MF 14327]

A new proof of the classical result concerning the cubic character of 2 (mod p) is outlined very briefly. *I. Niven* (West Lafayette, Ind.).

Chowla, S. Outline of a new method for proving results of elliptic function theory (such as identities of the Ramanujan-Rademacher-Zuckermann type). Proc. Lahore Philos. Soc. 7, 3 pp. (1945). [MF 15181]

It is pointed out that Ramanujan's identity

$$\sum p(5n+4)x^n=5\,\frac{(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\cdots}{(1-x)^6(1-x^2)^6(1-x^3)^6\cdots}$$

and other similar ones could be proved by a consideration of the algebraic differential equation satisfied by each member. *H. S. Zuckerman* (Seattle, Wash.).

Chowla, S. On the K-analogue of a result in the theory of the Riemann zeta function. Proc. Benares Math. Soc. 5, 23–27 (1943) = Proc. Lahore Philos. Soc. 6, no. 1, 9–12 (1944). [MF 14291]

The author proves that for any $\epsilon>0$ there exists a real primitive character χ (mod k), for some k, such that $L_\chi(1)<\epsilon$. Much more precise results were proved by Littlewood [Proc. London Math. Soc. (2) 27, 358–372 (1928)] on the assumption of the generalised Riemann hypothesis. Let $p_1, \cdots, p_g$ be the first g odd primes, $a=4p_1\cdots p_g$, b any integer with $(b/p_1)=\cdots=(b/p_g)=-1$, $b\equiv 1$ (mod 4). Let

$$S(x)=\sum_{n=1}^{x}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{m}{4an+b}\right),$$

where the symbol is the Jacobi symbol if m is odd and prime to $4an+b$, otherwise 0. The author proves that $\limsup_{x\to\infty} S(x)/x<\epsilon(g)$, where $\epsilon(g)\to 0$ as $g\to\infty$. He deduces his result from this by considering the values of n for which $4an+b$ is quadratfrei, so that the Jacobi symbol is a primitive character mod $2(4an+b)$. There are several misprints. *H. Davenport* (London).

Chowla, S. There exists an infinity of 3—combinations of primes in A. P. Proc. Lahore Philos. Soc. 6, no. 2, 15–16 (1944). [MF 14328]

The author proves that there exist infinitely many triplets of primes in arithmetic progression. *P. Erdös.*

★Finsler, Paul. Über die Primzahlen zwischen n und $2n$. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 118–122, Füssli, Zürich, 1945.

The author gives a simple proof of the theorem, first proved by Chebyshev, that there exists at least one prime between n and $2n$ for any integer $n\geqq 1$ ("Bertrand's postulate"). The proof is based on the complete factorization of the binomial coefficient $(2n)!/(n!n!)$ and is similar to that of Erdös [Acta Litt. Sci. Szeged 5, 194–198 (1932)]. By the same means he succeeds in showing that the number of primes between n and $2n$ is greater than $n/(3 \log 2n)$ and is less than $7n/(5 \log n)$. *H. W. Brinkmann.*

Archibald, Ralph G. Bertrand's postulate. Scripta Math. 11, 109–120 (1945). [MF 14660]

Expository article. The "postulate" states that if $n\geqq 1$ there is a prime p with $n<p\leqq 2n$.

Juzuk, D., and Tuchman, Z. Elementary bounds for the number of primes. Scripta Math. 11, 179–182 (1945). [MF 14662]

Elementary extensions of Euclid's proof of the infinitude of primes are given for primes in certain arithmetic progressions. For example, let $p_1, \cdots, p_\nu$ be the consecutive primes, not exceeding n, of the form $6k-1$ and let $P=p_1p_2\cdots p_\nu$. It is shown that there are at least $\nu-\mu+1$ primes of the form $6k-1$ between $n+1$ and $6P/p_\mu+p_\mu$, $1\leqq\mu\leqq\nu$, and that there are at least t primes of this form between $n+1$ and $6Pt-1$ for $t\leqq n+1$.

H. S. Zuckerman (Seattle, Wash.).

Bose, R. C., Chowla, S., and Rao, C. R. A chain of congruences. Proc. Lahore Philos. Soc. 7, no. 1, 53 (1945). [MF 15179]

It is announced without proof that a certain chain of congruences is solvable, the congruences being connected with the functional equation $g(2\cos\theta)=2\cos p\theta$ and the equation whose roots are $2\cos(a\pi/p)$, where a ranges over $1, 2, \cdots, p-1$. *I. Niven* (West Lafayette, Ind.).

Richmond, H. W. On the Diophantine equation $F\equiv ax^4+by^4+cz^4+dw^4=0$, the product $abcd$ being a square number. J. London Math. Soc. 19, 193–194 (1944). [MF 14562]

It is proved that one solution of the equation of the title can be used to generate a series of solutions. *I. Niven.*

Segre, B. On arithmetical properties of quadric and quartic surfaces. J. London Math. Soc. 19, 195–200 (1944). [MF 14563]

After obtaining explicitly all the quartic tetrahedral surfaces that are rational (that is, representable by an equation with rational coefficients), the author deduces two distinct methods for deriving new solutions from a known solution of quartic Diophantine equations of a certain type depending on seven parameters. One of these two methods includes that given by Richmond [see the preceding review]. One part of the proof depends on an arithmetical result for quadrics, not yet published. *I. Niven.*

Ljunggren, Wilhelm. A theorem of diophantine equations of the fourth degree. Avh. Norske Vid. Akad. Oslo. I. **1943**, no. 9, 13 pp. (1943). [MF 14158]

Let a, b, c, d be integers; a, b, c positive, $d\neq0$; a and c squarefree; d^2+4acb^2 not a perfect square. For every pair of natural numbers b_1, b_2 satisfying $b_1b_2=b$, the equations

$$(1)\qquad ab_1^2x^4-cb_2^2y^4-dx^2y^2=\pm1,$$
$$(1')\qquad a^2b_1^2x^4-c^2b_2^2y^4+acdx^2y^2=\pm1$$

are considered. If $a=c=1$, the equations (1′) are omitted, each of them being equivalent to an equation in the set (1). For fixed a, b, c, d, all equations in this set are insoluble in natural numbers x, y, except possibly one, which has exactly one solution. This is proved by showing that at most one unit of the field $K(\sqrt{a}, \sqrt{(-c)}, \theta)$, $\theta=\sqrt{(d+2b\sqrt{(-ac)})}$, has the form $p\sqrt{a}+q\sqrt{(-c)}+(s+t\sqrt{(-ac)})\theta$, where p, q, s, t are nonnegative rational integers, and $st=0$, $s+t\neq0$.

N. G. de Bruijn (Eindhoven).

Ljunggren, Wilhelm. Sur la résolution de quelques équations diophantiennes cubiques dans certains corps quadratiques. Avh. Norske Vid. Akad. Oslo. I. **1943**, no. 14, 23 pp. (1944). [MF 14157]

The following theorem is proved. Let $D>1$ be a rational integer other than 3, such that neither the number of ideal classes of $K(\sqrt{D})$ nor of $K(\sqrt{-D})$ is congruent to 0 (mod 3). Let $E>1$ be the fundamental unit of $K(\sqrt{D})$, and let $s=0$ or 1. Then the numbers of solutions of

$$\eta^2+1=E^s\xi^3,\qquad \xi\not\equiv0 \pmod 2,$$
$$\eta^2+1=2E^s\xi^3,\qquad \xi\not\equiv0 \pmod 2,$$
$$\eta^2+1=4E^s\xi^3$$

in integral numbers ξ, η of $K(\sqrt{D})$ are finite, and upper bounds can be found for the numbers of solutions. In some special cases the author obtains more precise results.

N. G. de Bruijn (Eindhoven).

Guinand, A. P. Gauss sums and primitive characters. Quart. J. Math., Oxford Ser. 16, 59–63 (1945). [MF 14166]

By means of an extension of the Poisson summation formula to sums involving primitive characters, due to Mordell [Proc. Cambridge Philos. Soc. 24, 585–596 (1928)], the author derives formulae which are analogous to the reciprocity formula for Gauss sums. Thus he shows that

$$\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk)$$
$$=e^{\frac14\pi i}(q/pk)^{\frac12}\tau(\chi)\sum_{n=1}^{pk}\bar\chi(n)\exp(-n^2\pi iq/pk),$$

where $\chi(n)$ is a primitive character modulo k, $\chi(-1)=1$, and $\tau(\chi)=\sum_{n=1}^{k}\chi(n)e^{2\pi in/k}$. A similar but more complicated result (with $\chi(n)$ replaced by $n\chi(n)$) holds for a primitive character $\chi(n)$ for which $\chi(-1)=-1$. Some elementary properties of the sums concerned are also given.

R. A. Rankin (Cambridge, England).

Mahler, K. On a theorem of Minkowski on lattice points in non-convex point sets. J. London Math. Soc. 19, 201–205 (1944). [MF 14564]

A star body K in n-dimensional Euclidean space is a closed bounded point set which contains the origin O as an inner point and whose frontier is met by every radius vector in just one point. Denote by V the volume of K and put $E=2$ if K is symmetrical with respect to O, $E=1$ otherwise. The following theorem was stated by H. Minkowski [Gesammelte Abhandlungen, vol. I, Teubner, Leipzig-Berlin, 1911, pp. 265, 270, 277] and proved for the first time by E. Hlawka [Math. Z. 49, 285–312 (1943); these Rev. 5, 201]. Let $n>1$, $V<E\zeta(n)$; then there exists a lattice of determinant 1 which contains no point of K other than O. The author presents a simple geometrical proof of the less precise statement where $\zeta(n)$ is replaced by n^{-1}.

C. L. Siegel (Princeton, N. J.).

Pipping, Nils. Zur Geometrie der Zahlen. Acta Acad. Aboensis 14, no. 13, 8 pp. (1944). [MF 15102]

If S_1, S_2, S_3 are the successive minima associated with a convex body of volume J in 3-dimensional space, a simple proof is given that $S_1S_2S_3J\leqq3!2^3$, an inequality that is weaker than Minkowski's [see Geometrie der Zahlen, Teubner, Berlin, 1910, chapter 5]. *H. Davenport* (London).

Ballieu, Robert. Sur des suites de nombres liées à une fraction continue régulière. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 165–174 (1943). [MF 13841]

Let $A=(a_0, a_1, a_2, \cdots)$ be a regular continued fraction, and put $A_i=(a_i, a_{i+1}, a_{i+2}, \cdots)$, $A_i'=(a_i, a_{i-1}, \cdots, a_1)$. The author investigates the limit points of the sets $\{A_i\}$ and $\{A_i'\}$, for example, when only a finite number of such limit

points exist, or when the finite limit points have periodic continued fractions. *K. Mahler* (Manchester).

Ballieu, Robert. Sur les suites périodiques. Mathesis **54**, 366–367 (1943). [MF 15533]

If the sequence $\{a_i\}$ admits a period h and $a_i = a_{ik}$ for $i \leq h-1$ then $\{a_i\}$ admits a period (h, k).

I. Kaplansky (Chicago, Ill.).

***Linés Escardó, Enrique. Aplicaciones de la Teoria de Redes Regulares al Estudio de las Funciones Cuasi-periodicas. [Applications of the Theory of Regular Nets to the Study of Quasiperiodic Functions].** Consejo Superior de Investigaciones Cientificas, Madrid, 1943. 79 pp. (Spanish)

Let $f(x_0, \cdots, x_n)$ be a real continuous function of period 1 in each of the variables $x_0, \cdots, x_n$; let the real numbers $\alpha_0, \cdots, \alpha_n$, 1 be linearly independent over the rational field; and let $k_0, \cdots, k_n$ be arbitrary real numbers. The author shows by means of Kronecker's theorem (for which a proof is given) that $\varphi(t) = f(\alpha_0 t + k_0, \cdots, \alpha_n t + k_n)$ is an almost periodic function of t, and that its Fourier series is

$$\varphi(t) \sim \sum_{\mu_0, \cdots, \mu_n} A_{\mu_0, \ldots, \mu_n} \exp\{2\pi i(\mu_0 k_0 + \cdots + \mu_n k_n)\} \times \exp\{2\pi i(\mu_0 \alpha_0 + \cdots + \mu_n \alpha_n)t\},$$

where $A_{\mu_0, \ldots, \mu_n}$ are the Fourier coefficients of $f(x_0, \cdots, x_n)$. Hence the characteristic exponents of $\varphi(t)$ form a module of finite base. The author deduces properties of $\varphi(t)$ from those of $f(x_0, \cdots, x_n)$, and also shows how generalized almost periodic functions can be obtained when $f(x_0, \cdots, x_n)$ is subjected to less restrictive conditions. The monograph concludes with applications to special ergodic problems and with a constructive method, by means of continued fractions, for solving inequalities

$$|\xi_1 - a\eta| < \epsilon_1, \quad |\xi_2 - b\eta| < \epsilon_2, \cdots, |\xi_n - l\eta| < \epsilon_n$$

in integers $\xi_1, \cdots, \xi_n, \eta$. *K. Mahler* (Manchester).

Chabauty, Claude. Approximation par des nombres formés avec un nombre fini de facteurs premiers et arithmétique des suites récurrentes. C. R. Acad. Sci. Paris **219**, 17–19 (1944). [MF 14514]

The author sketches a proof of the following theorem. Let K be an algebraic field of degree n, X, Y integers in K, $\boxed{X, Y}$ the maximum of the absolute values of X, Y and their conjugates, $\theta \neq 0$ any algebraic number, $W(\xi)$ any valuation in $K(\theta)$ and c a positive number. Then

$$0 < W(X/Y - \theta) < \boxed{X, Y}^{-c}$$

has only a finite number of solutions X, Y for which the prime ideal factors of XY belong to a finite set. From this result, applications to recurrent sequences are made [cf. K. Mahler, Nederl. Akad. Wetensch., Proc. **38**, 50–60 (1935); **39**, 633–640, 729–737 (1936)]. *K. Mahler.*

Wada, Yoshio. On the Diophantine analysis of algebraic functions. Proc. Imp. Acad. Tokyo **20**, 561–563 (1944). [MF 14925]

The author states results concerning algebraic functions of a single variable which are analogues of the theorems of Thue and Siegel on approximation of algebraic numbers, the valuation of an algebraic number by its absolute value being replaced by the valuation of an algebraic function at a given point. Detailed proofs are promised for a later paper.

C. L. Siegel (Princeton, N. J.).

Despujols, Pierre. Norme de l'unité fondamentale du corps quadratique absolu. C. R. Acad. Sci. Paris **221**, 684–685 (1945). [MF 15157]

In the real quadratic field generated by $\sqrt{m}$ over the rational field, let t be the number of prime factors of the discriminant, and r the number of independent invariant ideal classes which are regular (that is, contain an invariant ideal). The author proves as easy consequences of theorems in Hilbert's Zahlbericht [Jber. Deutsch. Math. Verein. **4**, 175–546 (1894)] that the norm of the fundamental unit is $(-1)^{t-r}$ and that the field contains irregular invariant ideal classes if and only if m is of the form u^2+v^2 and contains at least three distinct prime factors. *G. Whaples.*

***Fueter, Rud. Über die Normalbasis in einem absolut Abelschen Zahlkörper.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 141–152, Füssli, Zürich, 1945.

The author gives a new proof that every absolute Abelian field of odd degree l, with discriminant prime to l, has a normal basis for its ring of integers. This proof is more direct than that of Hilbert's Zahlbericht [Jber. Deutsch. Math. Verein. **4**, 175–546 (1894)], since it does not make use of the fact that every absolute Abelian field is a subfield of a cyclotomic field. *G. Whaples* (Philadelphia, Pa.).

ANALYSIS

Ostrowski, A. Über algebraische Relationen zwischen unbestimmten Integralen. Experientia **1**, 117–118 (1945). [MF 15388]

The note communicates, without proof, the following theorem. Let $p_i(x)$, $i=0, \cdots, n$, be algebraic functions. For each i, let w_i be an integral of p_i. Suppose that there exists a relation $F(w_0, \cdots, w_n; x) = 0$, whose first member is a nonzero polynomial in the w_i with algebraic functions of x for coefficients. Then there exists a relation

$$c_0 w_0 + \cdots + c_n w_n = a(x),$$

where the c_i are constants, not all zero, and where $a(x)$ is algebraic. An extension to integrals of functions belonging to any differential field is indicated. *J. F. Ritt.*

Ostrowski, A. Ein Unabhängigkeitssatz für irreduzible Integrale. Experientia **1**, 195 (1945). [MF 15389]

Let $\varphi_i(x)$, $i=1, \cdots, n$, be n functions which are elementary in the sense of Liouville. Let $\psi_i(x)$ be an integral of φ_i. It is announced that, if there exists a relation $F(\psi_1, \cdots, \psi_n, x) = 0$ with F an elementary function of its $n+1$ arguments, there exists a relation $c_1\psi_1(x) + \cdots + c_n\psi_n(x) = \psi(x)$, where the c_i are constants, not all zero, and where $\psi(x)$ is elementary.

J. F. Ritt (New York, N. Y.).

Ville, Jean. Sur l'opérateur exp $\{x+(d/dx)\}$. C. R. Acad. Sci. Paris **221**, 529–530 (1945). [MF 15139]

The author states some recursion formulas for the iterates of the operator in question and indicates an application to quantum mechanics. *H. Pollard* (New Haven, Conn.).

Ghabbour, M. N., and Winn, C. E. On the mode of approach of a repeated function to its limit. Proc. Math. Phys. Soc. Egypt 2, no. 1, 21–26 (1941). [MF 14110]

The authors obtain a variety of conditions on the positive continuous function $f(x)$ which imply that there is a $\Phi(n)$, independent of x, such that $f_n(x) \sim \Phi(n)$ as $n \to \infty$; here $f_1(x) = f(x)$, $f_n(x) = f(f_{n-1}(x))$, that is, $f_n(x)$ is the nth iterate of $f(x)$. Let $g(x) = 1 - f(x)/x$, $0 < g(x) < 1$, $g(0+) = 0$. One of the authors' results is the following. If $\limsup_{x\to 0} g(\lambda x)/g(x) < 1$, $0 < \lambda < 1$, while $\lim_{\lambda\to 1, x\to 0} g(\lambda x)/g(x) = 1$, then $f_n \sim \Phi(n)$, where $\Phi(n)$ is defined by $\int_{\Phi(n)}^{1} (tg(t))^{-1} dt = n$. *R. P. Boas, Jr.*

Wintner, Aurel. Remarks on the method of stationary phases. J. Math. Phys. Mass. Inst. Tech. 24, 127–130 (1945). [MF 15126]

Extract from the paper: "The principle of stationary phases states that, if $F(x)$ and $f(x)$ are sufficiently smooth, then (i) the asymptotic behavior of

$$\int_a^b F(x) e^{itf(x)} dx$$

as $t \to \infty$ depends only on the contribution of the immediate vicinity of those points . . . at which $f(x)$ becomes stationary . . . (ii) the asymptotic contribution of the vicinities of these points x_0 can be obtained by replacing the functions $F(x)$ and $f(x)$ by their respective approximations, $F(x_0)$ and $f(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$, near x_0." The author goes on to give precise mathematical conditions under which this principle is sound. *H. Pollard* (New Haven, Conn.).

Beckenbach, E. F. On a characteristic property of linear functions. Bull. Amer. Math. Soc. 51, 923–930 (1945). [MF 14462]

A proof of the theorem that, if $g(x)$ is continuous in a closed interval, and $2g(x_0) = g(x_0 - h_0) + g(x_0 + h_0)$ for each x_0 for some h_0 depending on x_0, then $g(x)$ is linear. The proof is in such a form that it holds when linear functions are replaced by more general classes of functions.

P. Franklin (Cambridge, Mass.).

Dieudonné, Jean. Dérivées et différences des fonctions de variables réelles. Ann. Sci. École Norm. Sup. (3) 61, 231–248 (1944). [MF 14650]

The author defines

$$\Delta f(x_0; h_1) = f(x_0 + h_1) - f(x_0),$$

$$\Delta^n f(x_0; h_1, \cdots, h_n) = \Delta^{n-1} f(x_0 + h_n; h_1, \cdots, h_{n-1}) - \Delta^{n-1} f(x_0; h_1, \cdots, h_{n-1}).$$

His principal result is that

$$\lim \Delta^n f(x_0; h_1, \cdots, h_n)/(h_1, \cdots, h_n) = a$$

exists as $h_1, \cdots, h_n \to 0$ if and only if the derivative $f^{(n-2)}(x)$ exists in a neighborhood of x_0 and $f^{(n-1)}(x_0)$ exists, $f^{(n-2)}(x)$ is the integral of a bounded function $u(x)$, and $\{u(x_0 + h) - u(x_0)\}/h \to a$ as $h \to 0$ with $x_0 + h$ in the set H where the derivative of $f^{(n-2)}(x)$ is $u(x)$. The ratio $\Delta^n f(x_0; h_1, \cdots, h_n)/(h_1, \cdots, h_n)$ is nonnegative for all small $h_1, \cdots, h_n$ if and only if $f^{(n-2)}(x)$ exists in a neighborhood of x_0 and has a right hand derivative $u(x)$, which is of bounded variation and has $\Delta^2 u(x_0; h_1; h_2)/h_1 h_2 \geqq 0$ for small h_1 and h_2. The results are generalized to functions of several real variables. [The principal interest of these results is that they localize the differential properties which positive or bounded differences of a given order imply. The author's corollaries concerning differential properties in a whole interval are essentially contained in previous work of Popoviciu [Mathematica, Cluj 8, 1–85 (1934)] and Boas and Widder [Duke Math. J. 7, 496–503 (1940); these Rev. 2, 219], who used differences with equally-spaced points. Cf. also Chlodovsky, C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 620–622 (1945); these Rev. 7, 149.] *R. P. Boas, Jr.*

Rasch, G. A synthesis of some interpolation, quadrature and summation formulas. Mat. Tidsskr. B. 1945, 129–141 (1945). (Danish) [MF 14266]

The author observes that many familiar formulas can be written in the form

$$(*) \qquad \int_{-\infty}^{\infty} f(x) d\mu(x) + \sum_{s=0}^{m} \int_{-\infty}^{\infty} f^{(s)}(x) d\lambda_s(x) + R_m,$$

$$R_m = \int_{-\infty}^{\infty} f^{(m+1)}(x) \nu_m(x) dx.$$

By assuming that (*) represents various classes of functions, such as polynomials and $(m+1)$th integrals of step functions, he discusses properties of and relations among the auxiliary functions entering in (*). Special cases include Taylor's formula and the Euler-Maclaurin summation formula. *R. P. Boas, Jr.* (Providence, R. I.).

Fourier Series and Generalizations, Integral Transforms

Prasad, B. N. The summability of a Fourier series and its conjugate series. Proc. 32nd Indian Sci. Congress, Nagpur 1945, part II, 24 pp. (1945). [MF 15374]

Presidential address to the Section of Mathematics and Statistics. There is a bibliography of 175 papers.

Wang, Fu-Traing. Note on H_2 summability of Fourier series. J. London Math. Soc. 19, 208–209 (1944). [MF 14568]

Let $f(x) \varepsilon L$ be of period 2π, let $s_n(x)$ be the partial sums of the Fourier series of f, and let $f(x_0 + t) + f(x_0 - t) - 2f(x_0) = \varphi(t)$. Hardy and Littlewood showed [Fund. Math. 25, 162–189 (1935)] that the condition $\int_0^t |\varphi(u)| du = o(t)$ does not imply

$$(*) \qquad \sum_0^n |s_\nu(x_0) - f(x_0)|^2 = o(n)$$

(strong summability of the Fourier series of f). Here the author proves that, if $\int_0^t |\varphi(u)| du = o(t |\log t|^{-\alpha})$ for some $\alpha > \frac{1}{2}$, then (*) does hold. *A. Zygmund.*

Kawata, Tatsuo. Notes on Fourier series. XII. On Fourier constants. Proc. Imp. Acad. Tokyo 20, 218–222 (1944). [MF 14887]

[Note XI, by Izumi and Kawata, appeared in Tôhoku Math. J. 47, 14–17 (1940); these Rev. 2, 94.] For a given sequence $a_1, a_2, \cdots$, form the sums $A_n = \sum_n^\infty a_k/k$. Bellman showed [Bull. Amer. Math. Soc. 50, 741–744 (1944); these Rev. 6, 125] that, if the a_n are the Fourier cosine coefficients of a function $f(x) \varepsilon L^p$, $1 < p$, then so are the A_n. The present paper gives the corresponding result for Fourier sine coefficients. It is also shown that, if $a_n > 0$, a necessary and sufficient condition for the A_n to be Fourier sine coefficients is that $\sum_2^\infty a_k \log k/k < \infty$. *P. Civin* (Buffalo, N. Y.).

Sunouchi, Gen-ichirô. On Fourier constants. Proc. Imp. Acad. Tokyo **20**, 542–544 (1944). [MF 14921]

The author obtains [independently] the result of Bellman quoted in the preceding review. *P. Civin.*

Kawata, Tatsuo. Notes on Fourier series. XIII. Remarks on the strong summability of Fourier series. Proc. Imp. Acad. Tokyo **20**, 223–226 (1944). [MF 14888]

[Cf. the second preceding review.] Let $s_n(x)$ be the partial sum of order n of the Fourier series of a function $f(x)\epsilon L$ and let $\theta_n(x)$ be the Rogosinski sum,

$$\theta_n(x)=\tfrac{1}{2}\{s_n(x+\pi/2n)+s_n(x-\pi/2n)\}.$$

Then, if

$$\max_{0<h\leq\delta}\int_0^{2\pi}|f(x+h)-f(x)|dx=O(1/|\log\delta|^{1+\epsilon}),\qquad \epsilon>0,$$

it has been proved [see Salem, C. R. Acad. Sci. Paris **208**, 70–72 (1939)] that the series $\sum_1^\infty|s_n(x)-\theta_n(x)|/n$ converges almost everywhere. The author gives a new proof of this result. Furthermore, he proves that, under the same assumptions, the series $\sum_1^\infty|s_n(x)-\sigma_n(x)|/n$ converges almost everywhere, $\sigma_n(x)$ being the $(C,1)$ means of the Fourier series. *R. Salem* (Cambridge, Mass.).

Sunouchi, Gen-ichirô. On the strong summability of Fourier series. Proc. Imp. Acad. Tokyo **19**, 420–423 (1943). [MF 14835]

It is known [see A. Zygmund, Fund. Math. **30**, 170–196 (1938)] that, if $f(x)$ is of period 2π and of the class L^p, $p>1$, then

$$(*)\qquad \int_0^{2\pi}\left\{\sum_{\nu=1}^\infty \nu^{-1}|s_\nu-\sigma_\nu|^2\right\}^{\frac{1}{2}p}dx\leq A_p\int_0^{2\pi}|f|^p dx.$$

The author shows that the exponent 2 on the left can be replaced by any number $m\geq 2$. By repeated application, an analogous result is obtained for multiple Fourier series. In particular, it is shown that, if $f(x,y)$ is of period 2π in x and y, and of the class L^p, $p>1$, then, denoting by $s_{n,n}(x,y)$ the square partial sums of the Fourier series of $f(x,y)$ we have

$$(n+1)^{-1}\sum_{\nu=0}^{n}|s_{\nu,\nu}-f|^m\to 0\qquad (n\to\infty)$$

at almost every point (x,y). [See also Marcinkiewicz, Ann. Scuola Norm. Super. Pisa (2) **8**, 149–160 (1939); Hsü, Bull. Amer. Math. Soc. **51**, 700–713 (1945); these Rev. **7**, 60.] *A. Zygmund* (Philadelphia, Pa.).

Chu, Liang-Pi. On the general partial sums of a Fourier series. Ann. of Math. (2) **46**, 511–532 (1945). [MF 14116]

The paper contains a number of results concerning various types of summability of Fourier series (Riesz means, Cesàro means, Hardy's series). They are too technical to be described in detail. *R. Salem* (Cambridge, Mas.).

Hardy, G. H., and Rogosinski, W. W. Notes on Fourier series. III. Asymptotic formulae for the sums of certain trigonometrical series. Quart. J. Math., Oxford Ser. **16**, 49–58 (1945). [MF 14165]

[Notes I and II appeared in J. London Math. Soc. **18**, 50–57, 83–87 (1943); these Rev. **5**, 65, 118.] The authors obtain asymptotic results on the behavior of the series $\sum a_n\cos nx$, $\sum b_n\sin nx$, $x\to 0$, when a_n and b_n behave like a power $n^{-\alpha}$, $\alpha=\beta+i\gamma$. The results are fairly complicated, and we state only the result for a special case: if $0<\beta<1$, $b_n=n^{-\alpha}$, then the series converge and

$$\sum b_n\sin nx\sim\xi b_\xi\Gamma(1-\alpha)\cos\tfrac{1}{2}\alpha\pi,\qquad x\to 0+,$$

where $\xi=[x^{-1}]$. There is an extensive bibliography on the subject, and extensions to Fourier integrals are indicated. *H. Pollard* (New Haven, Conn.).

Titchmarsh, E. C. On expansions in eigenfunctions. VII. Quart. J. Math., Oxford Ser. **16**, 103–114 (1945). [MF 14170]

[Part VI appeared in the same J. **12**, 154–166 (1941); these Rev. **3**, 235.] Continuing a study made in part V [same J. **12**, 33–50 (1941); these Rev. **3**, 121] the author studies expansions in the eigenfunctions of an equation

$$(*)\qquad y''-\{q(x)-\lambda\}y=0,\qquad 0\leq x<\infty.$$

The first part of the paper is devoted to a review and revision of the earlier paper. The latter part proves that, if $q(x)$ is any function which tends steadily to infinity, $f(x)$ belongs to $L^2(0,\infty)$, and the eigenvalues, eigenfunctions and Fourier coefficients are λ_n, $\psi_n(x)$ and c_n, respectively, then the expansion of $f(x)$ is summable in the following sense:

$$\lim_{\nu\to\infty}\sum_{n=0}^\infty\{\nu/(\nu+\lambda_n)\}c_n\psi_n(x)=f(x)$$

at almost all x, including the points of continuity. The proof depends on lemmas, of considerable interest in themselves, on the behavior of solutions of the equation $y''=p(x)y$, where $p(x)>0$. *H. Pollard* (New Haven, Conn.).

Titchmarsh, E. C. On expansions in eigenfunctions. VIII. Quart. J. Math., Oxford Ser. **16**, 115–128 (1945). [MF 14171]

In part VII [see the preceding review] the author established the summability of the expansion of a function $f(x)$ in $L^2(0,\infty)$ in terms of eigenfunctions of equation $(*)$ of the preceding review under very mild conditions on $q(x)$. He now proves that, under more severe restrictions on $q(x)$, namely, $q'(x)>0$, $q''(x)>0$ for $x>0$, $q''(x)=O[\{q'(x)\}^{14/13}]$, we actually have convergence almost everywhere. The procedure is to begin with the earlier result on summability and then to show that the additional restrictions enable one to invoke Wiener's Tauberian theorem to obtain convergence.

[The author illustrates his results [p. 118] by choosing $q(x)=x^2$ and remarks that the solutions $\psi_n(x)$ corresponding to $\lambda_n=2n+1$ are the normalized Hermite functions. This set of functions does not correspond to an initial condition problem on $(0,\infty)$, for they are orthogonal on $(-\infty,\infty)$. The statement should, therefore, be corrected to read that the solutions corresponding to $\lambda_n=4n+3$ are the odd Hermite functions, or, alternatively, those corresponding to $\lambda_n=4n+1$ are the even ones. These two sets arise from different initial conditions. There is a slight error in the first footnote of page 119, where the reference should be replaced by "Szegö, loc. cit., 194."] *H. Pollard.*

Andersson, W. The binomial type of Gram's series. Skand. Aktuarietidskr. **24**, 203–213 (1941). [MF 14133]

Gram's series for a function $f(x)$ of the integral variable x is obtained formally as follows. Let $\pi(x)$ be nonnegative and define orthonormal polynomials by

$$\sum_{x=-\infty}^{\infty}\pi(x)\Phi_\nu(x)\Phi_\mu(x)=\delta_{\mu\nu}.$$

Then

$$f(x)=\pi(x)\sum_{\nu=0}^{\infty}\gamma_\nu\Phi_\nu(x),$$

where $g(x)=f(x)/\pi(x)=\sum_{\nu=0}^{\infty}\gamma_\nu\Phi_\nu(x)$. The author discusses in detail the case where $\pi(x)=\binom{n}{x}p^x(1-p)^{n-x}$, $0<p<1$.

R. P. Boas, Jr. (Providence, R. I.).

Andersson, Walter. Short notes on Charlier's method for expansion of frequency functions in series. Skand. Aktuarietidskr. **27**, 16–31 (1944). [MF 14145]

This is a formal discussion of series of the forms (1) $f(x)=\sum_k a_k\phi^{(k)}(x)$ (continuous variable) and (2) $f(x)=\sum_k a_k\nabla^k\chi(x)$ (integral variable). For (1), the a_k are obtained from the power series of the quotient of the bilateral Laplace transform of $f(x)$ by that of $\phi(x)$. For (2), the Laplace transform is replaced by $\sum_{x=-\infty}^{\infty}(1+t)^x f(x)$. The author observes that if the generalization of (1), $F(x)=\sum a_k L^k\phi(x)$, where L is a differential operator, is applied to the Laplace transform of $f(x)$, we obtain an expansion $f(x)=\Phi(x)\sum a_k\{\psi(x)\}^k$, a type considered by Abel.

R. P. Boas, Jr. (Providence, R. I.).

Andersson, W. On the Gram series on Pearson's system of frequency functions. Skand. Aktuarietidskr. **25**, 141–149 (1942). [MF 14137]

The differential equation of Pearson's system of frequency curves,

$$\pi'(x)/\pi(x)=(a+x)/(b_0+b_1x+b_2x^2),$$

is used to give a general form to Romanovsky's developments. The series considered is $f(x)=\pi(x)\{1+\sum_{\nu=1}^{\infty}\gamma_\nu R_\nu(x)\}$, where

$$\int\pi(x)R_\nu(x)R_\mu(x)dx=\begin{cases}0, & \nu\neq\mu,\\ H_\mu, & \nu=\mu,\end{cases}$$

and $\gamma_\nu=(1/H_\nu)\int f(x)R_\nu(x)dx$. The author develops a recurrence formula of the type $R_{\nu+2}(x)-k_1R_{\nu+1}(x)+k_0R_\nu(x)=0$, where the k's are functions of ν, b_0, b_1, b_2, and a. If

$$R_\nu(x)=\sum_{i=0}^{\nu}\epsilon_{\nu,\nu-i}x^{\nu-i},$$

it is shown that

$$b_0(\tau+1)(\tau+2)\epsilon_{\nu,\tau+2}+(\tau+1)\{a+(\tau+1)b_1\}\epsilon_{\nu,\tau+1} \\ -(\nu-\tau)\{1+(\nu+\tau+1)b_2\}\epsilon_{\nu,\tau}=0,$$

where $\epsilon_{\nu,\nu}=1$ and $\epsilon_{\nu,\nu-1}=\nu(a+\nu b_1)/(1+2\nu b_2)$. Formulas are given for $R_0(x),\cdots,R_4(x)$.

R. L. Anderson.

✶Carleman, T. L'Intégrale de Fourier et Questions que s'y Rattachent. Publications Scientifiques de l'Institut Mittag-Leffler, 1. Uppsala, 1944. 119 pp.

This monograph contains a summary of lectures by the author and certain additional material. The exposition is clear and is well motivated throughout. Chapter I contains a derivation of the Fourier integral and Plancherel theorems. Chapter II is on the application of analytic functions to the Fourier transform. This chapter generalizes the fact that any function $f(x)$ representable by a Fourier integral can be written as

$$f(x)=\int_{-\infty}^{0}g(u)e^{iux}du+\int_{0}^{\infty}g(u)e^{iux}du,$$

where the first integral represents a function analytic in the lower half complex x-plane and the second analytic in the upper half plane. [The author appears to be unfamiliar with the far more extensive technique of the Fourier transform in the complex domain as developed by Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., vol. 19, New York, 1934].] Chapter III is on systems of integral equations and related problems in approximation. It is used subsequently in proving Wiener's theorem on the closure in L_1 of the translations of a function whose Fourier transform does not vanish. The technique of the Fourier transform for solving integral equations is mentioned only in the appendix with reference to a degenerate case. Chapter IV is on the Fourier transforms of absolutely integrable functions. Rather than dealing with the transform of the reciprocal of the function as in Wiener's original result, the author deals with an analytic function of the integrable function as in Lévy's generalization. Chapter V is on Wiener's Tauberian theorem. [The proof of the Wiener-Ikehara theorem does not make use of the simplifications introduced by Landau, Heilbronn and others.] Chapter VI gives an account of certain aspects of the spectrum of a function of a real variable.

N. Levinson (Cambridge, Mass.).

Yosida, Kôsaku. On the representation of functions by Fourier integrals. Proc. Imp. Acad. Tokyo **20**, 655–660 (1944). [MF 14939]

Six theorems about positive definite functions and Fourier transforms are stated for the real line but it is asserted they can be extended to locally compact Abelian groups because their proofs involve only the Plancherel theorem. Let $f(t)$ be a continuous bounded complex-valued function defined for $-\infty<t<\infty$ and consider the following expressions:

(a) $$\varphi_n(\lambda)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}f(t)(t/n)^{-2}\sin^2(t/n)e^{-i\lambda t}dt,$$

(b) $$f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda).$$

We state three of the theorems: (1) $f(t)$ is positive definite if and only if $\varphi_n(\lambda)\geqq 0$ for $n=1,2,\cdots$; (2) $f(t)$ is representable in the form (b) with $v(\lambda)$ of bounded variation if and only if the L_1-norms of the $\varphi_n(\lambda)$ are bounded; (3) for an $f(t)$ representable in form (b), $v(\lambda)$ is absolutely continuous if and only if the $\varphi_n(\lambda)$ converge in L_1 as $n\to\infty$, and in the case where $v(\lambda)$ is nondecreasing $v(\lambda)$ is singular if and only if $\varphi_n(\lambda)\to 0$ almost everywhere. [Some of these results are special cases of theorems of H. Cramér, Trans. Amer. Math. Soc. **46**, 191–201 (1939); these Rev. **1**, 13.]

W. Ambrose (Ann Arbor, Mich.).

Borgen, S. Note on Poisson's formula. J. London Math. Soc. **19**, 213–219 (1944). [MF 14570]

If $f(t)$ is continuous, belongs to $L^2(-\infty,\infty)$, and satisfies

$$\int_{-\infty}^{\infty}|f(t+h)-f(t-h)|^2dt=O(|h|^{1+\alpha}),\qquad h\to 0,$$

where $0<\alpha\leqq 1$, then, with $(2\pi)^{\frac{1}{2}}F(x)=\int_{-\infty}^{\infty}f(t)e^{ixt}dt$, Poisson's formula

$$(2\pi)^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}f(2\pi n)e^{2\pi nti}=\sum_{n=-\infty}^{\infty}F(t+n)$$

holds in the sense that both sides converge in mean square on $(0,1)$ to the same sum, while for almost all t the left side is summable $(C,1)$ and the right side converges to the same sum. This is no longer true when $\alpha=0$. A corollary is that the same hypotheses imply that $f(t)\to 0$ as $t\to\infty$ and that $\sum|f(n\lambda)|^2$ converges, $\lambda>0$.

R. P. Boas, Jr.

Kravtchenko, Julien. Sur l'extension d'un théorème de Fatou et Priwaloff. C. R. Acad. Sci. Paris **219**, 47–49 (1944). [MF 14511]

A theorem of Privaloff [Bull. Soc. Math. France **44**, 100–103 (1916)] asserts that, if $f(x)$ is of period 2π and of the class Lip α, $0<\alpha<1$, the conjugate function

$$\tilde{f}(x)=-\pi^{-1}\int_0^\pi \{f(x+t)-f(x-t)\}\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt$$

also belongs to Lip α. The author proves a more general result by considering the modulus of continuity of f. [The generalization is not new: see, for example, Zygmund, Prace Mat.-Fiz. **33**, 125–132 (1924).] *A. Zygmund.*

Humbert, Pierre. Sur les formules opératoires du calcul symbolique. C. R. Acad. Sci. Paris **221**, 398–399 (1945). [MF 14682]

Let $g(x)/x$ be the Laplace transform of $f(x)$. The problem of expressing in terms of $f(x)$ the Laplace transform of $g(x)$ or $g(h(x))$, where $h(x)$ is some preassigned function, is discussed. The results are all formal. *H. Pollard.*

Alessi, Juan M. The Le Roy-Stieltjes transform. An. Soc. Ci. Argentina **140**, 121–127 (1945). (Spanish) [MF 14096]

The transform is $f(s)=\int_0^\infty(1+ts)^{-1}d\alpha(t)$. The change of variable $w=1/z$ reduces it, as the author is aware, to the well-known Stieltjes transform. However, he develops a number of its properties directly, without using the Stieltjes transform. *R. P. Boas, Jr.* (Providence, R. I.).

Kovanko, A. Sur la compacité des systèmes de fonctions presque-périodiques généralisées de A. Besicovitch. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 365–382 (1945). (Russian. French summary) [MF 14589]

Detailed proofs of results previously announced [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 117–118 (1941); **43**, 49–50 (1944); these Rev. **3**, 107; **6**, 265].

Functional Analysis, Ergodic Theory

Iwamura, Tsurane. A generalized limit. Proc. Imp. Acad. Tokyo **20**, 346–347 (1944). [MF 14899]

Definition and existence proof for a generalization of the ordinary notion of limit in a locally bicompact Hausdorff space. *E. Hewitt* (Princeton, N. J.).

Nakano, Hidegorô. Über das System aller stetigen Funktionen auf einem topologischen Raum. Proc. Imp. Acad. Tokyo **17**, 308–310 (1941). [MF 14711]

Let R be any topological space and let $\{f_\lambda\}$ be any family of real-valued, continuous, nonnegative functions defined on R. It is proved that, if R is an extremal space (that is, if the closure of any open set is open), then the family $\{f_\lambda\}$ has a greatest lower bound f_0. If R is completely regular, then the condition is also necessary. An analogous theorem is proved for countable families of functions and normal spaces. [It is to be noted that f_0 need not be the point-wise infimum of the functions f_λ. If such a requirement is made, the situation becomes much more complex.] *E. Hewitt.*

Nakano, Hidegorô. Über normierte teilweisegeordnete Moduln. Proc. Imp. Acad. Tokyo **17**, 311–317 (1941). [MF 14712]

The author continues in this paper previously reported investigations on vector lattices [same Proc. **16**, 437–441 (1940); Proc. Phys.-Math. Soc. Japan (3) **23**, 485–511 (1941); these Rev. **2**, 343; **3**, 210]. He considers a vector lattice with a norm which need not satisfy the triangle inequality but which enjoys other strong properties. It is stated that, under one set of conditions, a normed vector lattice B may be represented as the set of continuous functions with integrable kth powers from a locally bicompact Hausdorff space X with measure to the extended real number system, where the norm in B is equal to a certain integral over X. Two similar representation theorems are stated for vector lattices with other types of norm. No detailed proofs are given. *E. Hewitt* (Princeton, N. J.).

Nakano, Hidegorô. Über die Charakterisierung des allgemeinen C-Raumes. Proc. Imp. Acad. Tokyo **17**, 301–307 (1941). [MF 14710]

Using a preliminary result announced in the paper reviewed above, the author proves the following theorem. Let M be a vector lattice over the real numbers with a norm such that (1) $\|a\|\geqq 0$, and $\|a\|=0$ if and only if $a=0$; (2) $\|\alpha a\|=|\alpha|\,\|a\|$ for every real number α; (3) $|a|\leqq|b|$ implies $\|a\|\leqq\|b\|$, where $|x|$ is defined as $(x\cup 0)+((-x)\cup 0)$; (4) $\|(|a|\cup|b|)\|=\max(\|a\|,\|b\|)$; (5) $\sup\|a_\gamma\|$ for $\gamma\epsilon\Gamma$ is equal to $\inf\|b\|$ for $b\geqq a_\gamma$, $\gamma\epsilon\Gamma$. Then M is linearly, lattice- and norm-isomorphic to the set of all real-valued continuous functions x on a bicompact Hausdorff space R, or to the set of all real-valued continuous functions on R which vanish at a certain point. The norm of x is $\sup|x(p)|$ for $p\epsilon R$. *E. Hewitt* (Princeton, N. J.).

Nakano, Hidegorô. Über die Charakterisierung des allgemeinen C-Raumes. II. Proc. Imp. Acad. Tokyo **18**, 280–286 (1942). [MF 14762]

A revised proof of the theorem stated in the preceding review. *E. Hewitt* (Princeton, N. J.).

Nakano, Hidegorô. Über die Stetigkeit des normierten teilweise geordneten Moduls. Proc. Imp. Acad. Tokyo **19**, 10–11 (1943). [MF 14792]

The author shows that, in a Banach lattice, if $a_n\downarrow O$ in order implies $\|a_n\|\downarrow 0$ in norm for ordinary sequences, then $a_\alpha\downarrow O$ in order implies $\|a_\alpha\|\downarrow 0$ in norm for all transfinite sequences as well. *G. Birkhoff* (Cambridge, Mass.).

Nakano, Hidegorô. Über ein lineares Funktional auf dem teilweise geordneten Modul. Proc. Imp. Acad. Tokyo **18**, 548–552 (1942). [MF 14779]

Let $\mathfrak{M}$ be a partially ordered module (real linear space). Let $\mathfrak{E}$ be the space of all maximal ideals of projectors in $\mathfrak{M}$ as defined and discussed in an earlier paper of the author [Proc. Phys. Math. Soc. Japan (3) **23**, 485–511 (1941); these Rev. **3**, 210]. In the present work the author defines special kinds of points in $\mathfrak{E}$ and uses the theory of these to show that the members of a certain class of partially ordered modules including the L^p spaces cannot be represented as modules of real functions. As the author observes, Nakayama was the first to prove the existence of such partially ordered modules. In addition, a theorem is proved giving a necessary and sufficient condition for a positive linear functional on the space of continuous functions on a bicompact Hausdorff space to be of the form $P(f)=f(p)$, where p is a point in the Hausdorff space. Some of the details in the paper are obscure to the reviewer because of the present unavailability of an earlier work of the author [J. Fac. Sci. Imp. Univ. Tokyo, Sect. I. **4**, 201–382 (1942)] in which the terminology is explained. *G. W. Mackey.*

Orihara, Masae. On the regular vector lattice. Proc. Imp. Acad. Tokyo 18, 525–529 (1942). [MF 14773]

Let L be a regular vector lattice in the sense of Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. 2(44), 121–165 (1937)]. The author proves that the regularity axiom for complete vector lattices is equivalent to the following two conditions: (I) $\lim_{k\to\infty} y_i^{(k)} = y_i$ in L and $\lim_{i\to\infty} y_i = y$ in L imply that there exists an increasing sequence of indices $k_1, k_2, \cdots$ such that $\lim_{i\to\infty} y_i^{(k_i)} = y$; (II) for any nonvoid subset E of L, there exists a denumerably infinite or finite subset E^* of L such that $\sup E^* = \sup E$. It is further shown that various results proved by Kantorovitch for regular vector lattices are true for complete vector lattices satisfying one of the conditions (I) and (II). *E. Hewitt.*

Matsuyama, Noboru. On locally convex topological spaces. Proc. Imp. Acad. Tokyo 18, 585–587 (1942). [MF 14782]

The author studies a linear space L with a triangular pseudo-norm $|x|_d$ defined for x in L and d in a directed system D. Hyers [Duke Math. J. 5, 628–634 (1939); these Rev. 1, 58] showed that such a space is a locally convex linear topological space. For every d in D and f in L^*, the space of distributive continuous functionals on L, let $|f|_d = \sup |f(x)|/|x|_d$ for $x \epsilon L$, and let L^*_d be the subspace of those f in L^* for which $|f|_d$ is finite. Then each L^*_d is a complete normed linear space. The conjugate space L^{**} of L^* is defined to be the set of all distributive functionals on L^* which are continuous on every L^*_d. Then for every X in L^{**}, $|X|_d = \sup |X(f)|/|f|_d$ for $f \epsilon L^*_d$ defines a triangular pseudo-norm in L^{**}. The natural mapping of L into L^{**}, defined by $x \to X$ if $X(f) = f(x)$ for all f in L^*, is a linear transformation preserving the pseudo-norm; hence the problem of reflexivity, so well known in normed spaces, arises again in spaces with a triangular pseudo-norm.

M. M. Day (Urbana, Ill.).

Nakamura, Masahiro. Note on Banach spaces. III. A proof of Tietze-Matsumura's theorem. Proc. Imp. Acad. Tokyo 18, 267–268 (1942). [MF 14759]

[Notes I and II, by Izumi and the author, respectively, appeared in the same Proc. 18, 127–130, 131 (1942). The notes of this series are not related.] Call a subset S of a linear topological space E locally convex if for every point p of S there is a neighborhood of p whose intersection with S is convex. It is known [H. Tietze, Math. Z. 28, 697–707 (1928); S. Matsumura (Nakajima), Tôhoku Math. J. 27, 227–230 (1928); the author's reference is incorrect] that any closed, connected, locally convex subset of Euclidean n-space is convex. The author extends this theorem to any uniformly convex normed E by requiring in addition that S is compact. [Reviewer's comments. (1) The author's proof applies word for word in a strictly convex, normed E. (2) Application of the finite dimensional theorem at an early stage of the author's proof shows that under all these restrictions on S the theorem holds in any linear topological E; the theorem also holds in any linear topological E if "compact and connected" is replaced by "arcwise connected."] *M. M. Day* (Urbana, Ill.).

Aoki, Tosio. Locally bounded linear topological spaces. Proc. Imp. Acad. Tokyo 18, 588–594 (1942). [MF 14783]

It is shown that the following are equivalent for a topological linear space. (1) The space contains a bounded open set. (2) The topology may be introduced by an "absolute value" having all the properties of a norm except that the triangle inequality is replaced by $|x+y| \leqq k|x| + k|y|$, where $k \geqq 1$ and is independent of x and y. (3) The topology may be introduced by an "F' norm" having all the properties of a norm except that the homogeneity is replaced by $\|\alpha x\| = |\alpha|^r \|x\|$, where $0 < r \leqq \log 2/\log 3$. In addition, the continuity of the absolute value is discussed. [The equivalence of (1) and (2) has also been proved by Bourgin [Amer. J. Math. 65, 637–659 (1943); these Rev. 5, 103]. It is a sharpening of a result of Hyers [Revista Ci., Lima 41, 555–576 (1939); these Rev. 1, 318].] *G. W. Mackey.*

La Salle, J. P. Application of the pseudo-norm to the study of linear topological spaces. Revista Ci., Lima 47, 545–563 (1945). [MF 15434]

As has been observed by Hyers [Duke Math. J. 5, 628–634 (1939); these Rev. 1, 58] it is possible to introduce the topology of a topological linear space by means of a suitable partially ordered family of real-valued functions on the space. Such a family he calls a pseudo-norm. The author modifies Hyers' procedure slightly and investigates the forms of the definition of limit and of continuous function, the condition that a nonzero linear functional exist, the condition that a linear transformation have a linear inverse, etc., when expressed in terms of the pseudo-norm. In addition he notes some special properties of the pseudo-norm when it is what he calls pseudo-convex. Finally he uses the pseudo-norm to give a definition of differential and shows that it has the usual properties. *G. W. Mackey.*

Milman, D. On the normability of topological rings. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 162–164 (1945). [MF 13736]

Using methods developed for fields by I. Shafarevitch [same C. R. (N.S.) 40, 133–135 (1943); these Rev. 6, 164] the author gives conditions under which a topological ring K may be normed. Let R be the set of all $x \epsilon K$ such that $\lim_{n\to\infty} x^n = 0$. Then a norm may be introduced in K if R is open, bounded, convex, and contains an element p which is nonsingular (p^{-1} exists). The proof uses the function $n(x)$ defined for each $x \epsilon K$ to be the largest integer for which $x \epsilon p^n \cdot R$. Then if $\alpha(x) = \lim_{k\to\infty} n(x^k)/k$, the norm $\|x\|$ is, up to a necessary refinement, defined to equal $\rho^{\alpha(x)}$, where $0 < \rho < 1$. Rings K are then studied which contain the rational or the rational complex numbers. Using the metric introduced into K, the ring is completed by means of fundamental sequences to an enveloping normed ring $\bar{K}$. Since $\|x^m\| = \|x\|^m$, $\bar{K}$ is a ring of continuous functions defined on a bicompact space $\mathfrak{M}$ (the space of maximal ideals in $\bar{K}$). Thus K is essentially a dense subring of the ring of all functions continuous on a bicompact space $\mathfrak{M}$ with the obvious topology. *E. R. Lorch* (New York, N. Y.).

Komatuzaki, Hitosi. Une remarque sur les projections dans certains espaces du type (*B*). Proc. Imp. Acad. Tokyo 17, 238–240 (1941). [MF 14706]

The author gives a direct and simpler proof of his earlier theorem [same Proc. 16, 274–279 (1940); these Rev. 2, 102] that there exists a closed subspace of the Banach space (c_0) on which there is no projection. His methods are closely parallel to those used by Murray [Trans. Amer. Math. Soc. 41, 138–152 (1937)] in establishing the corresponding theorem for the spaces (l^p) with $p > 1$. [It seems appropriate to point out that a paper by Sobczyk [Duke Math. J. 8, 78–106 (1941); these Rev. 2, 220] appeared at about the same time as this one, giving a very short and elegant proof of this theorem for both (c_0) and the (l^p) spaces.]

G. W. Mackey (Cambridge, Mass.).

Hanai, Sitiro. Extension of linear functionals on pseudo-normed linear spaces. Jap. J. Math. **18**, 973–975 (1943). [MF 14989]

The author gives a complicated proof of the fact that a linear functional on a subspace of a certain kind of metric linear space may be extended to the whole space so as to be additive and homogeneous. [Since he does not prove that the extension need be continuous (one may show, in fact, that his construction can lead to a discontinuous extension) this result is contained in the well-known and obvious theorem to the effect that an additive and homogeneous functional on a subspace of an arbitrary linear space may be so extended.] *G. W. Mackey* (Cambridge, Mass.).

Hanai, Sitiro. On biorthogonal systems in Banach spaces. Proc. Imp. Acad. Tokyo **20**, 510–512 (1944). [MF 14916]

Let $x_1, x_2, \cdots$ be a sequence of elements in a Banach space E. Theorem 1 states the [obvious] fact that there exists a sequence $f_1, f_2, \cdots$ in $\bar{E}$ such that $f_i(x_j)=\delta_i^j$ for $i, j=1, 2, \cdots$ if and only if no x_i is contained in the closed linear span of the rest. Theorem 2 states that theorem 1 holds when the roles of the x_i and f_i are interchanged. [It is easy to construct counterexamples showing that the nonobvious half of this theorem is false. The author's mistake consists in assuming that a decreasing sequence of closed subsets of a complete metric space necessarily has a nonempty total intersection. Theorem 3 is a statement of the true half of theorem 2 in a special case.]
G. W. Mackey (Cambridge, Mass.).

Nef, Walter. Lineare Funktionale in kompakten metrischen Räumen. Comment. Math. Helv. **17**, 214–220 (1945).

In an earlier paper [same Comment. **16**, 284–304 (1944); these Rev. **6**, 208] the author showed how one can define Stieltjes integrals on a compact metric space in a topologically invariant manner. He applied without proof a theorem related to such integrals. In the present paper he proves this theorem, which is as follows. Let M be a compact metric space, let $f_j(P)$ $(j=1, 2, \cdots)$ be denumerably many real continuous functions of M and let c_j be real constants. The system of equations

$$\int_M d[\theta(\mu)]f_j(P)=c_j, \qquad j=1, 2, \cdots,$$

has a solution $\theta(\mu)$ which is of bounded variation on M if and only if a positive constant F exists so that for every positive integral N and every set of N arbitrary real constants $\vartheta_1, \cdots, \vartheta_n$ the inequality

$$\left|\sum_{j=1}^{n}\vartheta_j c_j\right| \leqq F\cdot\max\left|\sum_{j=1}^{n}\vartheta_j f_j(P)\right|$$

holds. In such a case there exists a solution $\theta(\mu)$ whose total variation on M is less than or equal to F. This theorem is a generalization of a theorem of F. Riesz on functions defined on a real interval $a\leqq x\leqq b$ [C. R. Acad. Sci. Paris **150**, 674–677 (1910)]. The author obtains his results from Riesz's theorem by a continuous mapping of the real interval (a, b) onto M. The proof also makes use of notions introduced in the author's paper cited above. The paper closes with a proof of the theorem for quaternions. *W. T. Martin.*

Van der Lijn, G. Une généralisation de l'intégrale de Radon. Bull. Soc. Roy. Sci. Liége **10**, 168–175 (1941). [MF 13057]

The general integral $I(x, E)$ is defined as a limit element of a Banach space B in the following way. Let S be an arbitrary space of elements s, $\mathfrak{A}$ a ring of subsets of the space S, $\varphi(E)$ a finitely additive operator, defined for every set E of $\mathfrak{A}$, whose "value" φ is an element of the Banach space B. The total variation $V(E)$ of φ is the upper bound, finite or infinite, of $\sum_1^n\|\varphi(E_i)\|$ for all finite partitions of E into disjoint sets $E_1, \cdots E_n$ in $\mathfrak{A}$, $\|\varphi\|$ representing the norm of φ. It is assumed that $\varphi(E)$ is of bounded variation. Thus $0\leqq\|\varphi(E)\|\leqq V(E)$. Let $x(s)$ be a functional determined for every element s of E; it is associated with $\mathfrak{A}$ in the sense that the sets $E(\alpha)=E(x(s)\leqq\alpha)$ and $E(\alpha)=E(x(s)<\alpha)$ both belong to $\mathfrak{A}$ for each α. When $x(s)$ is bounded in E, $l<x(s)<L$, consider a ladder, $l_0=l<l_1<\cdots<l_n=L$, such that $l_i-l_{i-1}<\epsilon$, $i=1, \cdots, n$, and let $E_i=E[l_{i-1}<x(s)\leqq l_i]$. Since E_i belongs to $\mathfrak{A}$, the sum $\sigma=\sum_{i=1}^n l_i\varphi(E_i)$ is an element of B. The author proves that, as $\epsilon\rightarrow 0$, σ converges to a unique element in B. This limit element is, by definition, the generalized integral $I(x, E)$ of $x(s)$ on E. The author next establishes counterparts of four basic theorems for bounded functions: $I(x, A+B)=I(x, A)+I(x, B)$, $\|I(x, E)-I(y, E)\|\leqq\delta V(E)$ if $|x(s)-y(s)|\leqq\delta$, $I(x+y, E)=I(x, E)+I(y, E)$, $\|\int_E x(s)d\varphi(E)\|\leqq\int_E|x(s)|dV(E)$. Finally, summability of a functional $x(s)$ with respect to φ is defined by requiring that $|x(s)|$ is summable with respect to $V(E)$ in the usual sense; $I(x, E)$ is then the limiting element of B to which I_α^β converges when α, β independently tend to $-\infty, \infty$, respectively, I_α^β being the integral of the bounded function $[x(s)]_\alpha^\beta$ bounded below and above by α and β, respectively. The four theorems stated above are valid for summable functions also. *H. E. Bray* (Houston, Tex.).

Zorn, Max A. Characterization of analytic functions in Banach spaces. Ann. of Math. (2) **46**, 585–593 (1945). [MF 14122]

The author proves that, if $f(x)$ is a function from one complex Banach space to another, such that

$$\lim_{\zeta\rightarrow 0}[f(x+\zeta h)-f(x)]/\zeta$$

exists for x in an open set around the origin and h arbitrary and $f(x)$ is continuous when a suitable set of the first category has been removed from its domain, then $f(x)$ has a power series expansion and a Fréchet differential. By considering the linear functionals on the range space, the general case is reduced to that in which $f(x)$ is numerically valued and for this the author uses theorems on functions of a number of complex variables. Further results on sum spaces are also given. *F. J. Murray* (New York, N. Y.).

Shimoda, Isae. On analytic functions in abstract spaces. Proc. Imp. Acad. Tokyo **19**, 462–465 (1943). [MF 14843]

The paper contains four theorems about analytic functions which map a region in one complex Banach space E into another complex Banach space E'. The point of departure is the expansion of such a function in a series of homogeneous polynomials [for references to the literature see A. E. Taylor, Bull. Amer. Math. Soc. **49**, 652–669 (1943); these Rev. **5**, 39]. The main theorem is the following. Let $y=f(x)$ map the interior of the unit sphere in E in one to one fashion on the interior of the unit sphere in E'; let the direct mapping and its inverse both be analytic and let $f(0)=0$. Then $f(x)$ is linear and the mapping is isometric, that is, norm preserving. The proof rests in part on the following theorem. If $f(x)$ is analytic in the unit sphere of E, with values in this same sphere, and if $f(x)$ differs from x by a series of homogeneous polynomials of degree not less than 2, then $f(x)\equiv x$. *A. E. Taylor* (Los Angeles, Calif.).

Julia, Gaston. Les projections des systèmes orthonormaux de l'espace hilbertien et les opérateurs bornés. C. R. Acad. Sci. Paris **219**, 8–11 (1944). [MF 14516]

The author considers a linear closed manifold h in a Hilbert space H and a sequence A_i of elements in h. He seeks characterizing conditions that this sequence be the projection on h of an orthonormal sequence E_i. He further wishes to determine the E_i if they exist. Assume that h and its complement are of infinite dimensionality and determined by the orthonormal bases e_k, ϵ_k. Let the linear closed manifold determined by E_i be V. The operators $Ax=\sum x_i A_i$ and $Bx=\sum x_i B_i$ define linear bounded operators on h and its complement, respectively. It is then shown that $M_A \leqq 1$ (M_A, the bound of A) is necessary and sufficient for the existence of the E_i such that the A_i are the projections on h of E_i. If this condition is satisfied the operator B_i may be taken as $JH\epsilon_i$, where J is an arbitrary isometric operator and $H=[I-A^*A]^{\frac{1}{2}}$. The author then gives a characterization of A_i in order that E_i be complete: (1) the linear closed domain determined by $A_1, A_2, \cdots$ must be h; (2) if $w=A^*(h)$, $A^*A=P_w$, the projection operator; (3) $(h-w)$ must be of infinite dimensionality. *H. H. Goldstine.*

Julia, Gaston. Les symétries dans l'espace hilbertien. C. R. Acad. Sci. Paris **221**, 81–83 (1945). [MF 14239]

Just as the properties of being Hermitian and idempotent characterize orthogonal projection operators, the author shows that the properties of being Hermitian and unitary characterize the symmetry (orthogonal reflection) operators. If $\mathfrak{M}$ is a closed linear manifold, its symmetry operator S has $\frac{1}{2}(f+Sf)$ lying in $\mathfrak{M}$ and $f-Sf$ orthogonal to $\mathfrak{M}$ for all f. If E is the orthogonal projection on $\mathfrak{M}$, then $S=2E-1$. Simple properties of symmetries and their products are treated. It is shown that $f_1, \cdots, f_n$ can be mapped into $g_1, \cdots, g_n$, respectively, by a symmetry if and only if, for all i, j, $(f_i, f_j)=(g_i, g_j)$ and $(f_i, g_j)=(g_i, f_j)$.
I. Halperin (Kingston, Ont.).

Julia, Gaston. Décomposition des opérateurs unitaires ou isométriques en produit de symétries. C. R. Acad. Sci. Paris **221**, 265–267 (1945). [MF 14495]

If $\varphi_1, \varphi_2, \cdots$ is an arbitrary complete, orthonormal basis in a unitary space and V is an everywhere defined isometric operator, then V can be expressed by products $V=\{\lim_{n\to\infty}(S_1 \cdots S_n)\}U$. Here the S's are suitable symmetry operators [see the preceding review], the product converges strongly, and U is a unitary operator with the φ's as spectral vectors, that is, $U\varphi_m=C_m\varphi_m$, $|C_m|=1$, for all m. If V has a real matrix in terms of the φ's, then the factor U can be omitted and the S's will also have real matrices. *I. Halperin* (Kingston, Ont.).

Julia, Gaston. Sur deux propriétés des matrices infinies, hermitiennes positives. C. R. Acad. Sci. Paris **221**, 317–319 (1945). [MF 14502]

The author examines two theorems. (i) If $\varphi_1, \varphi_2, \cdots$ is an arbitrary complete orthonormal set in a Hilbert space and H is a bounded positive-definite operator, then $H=TT^*$ where the matrix of T, in terms of the φ's, is "recurrent," that is, has 0's above the main diagonal. His comment amounts to this: if $H=SS^*$, where S need not be recurrent, then $T=SU$, where the unitary operator U is defined by $U\varphi_m=\psi_m$, the ψ's being obtained from the $S^*\varphi$'s by application of the Schmidt orthonormalization process. Conversely, starting from $H=SS^*$, say with $S=S^*$, the positive square root of H, and defining U as above, he derives the recurrent decomposition.

(ii) Vectors f_i can be found in a Hilbert space to satisfy a set of equations

$$(*) \qquad (f_i, f_j)=a_{ij}, \quad i, j=1, 2, \cdots; a_{ij}=\bar{a}_{ji},$$

if and only if $G_n=\sum_{i,j=1}^{n} a_{ij}x_i\bar{x}_j$ is a nonnegative form for all n. He points out that the dimension of the manifold spanned by $f_1, f_2, \cdots$ is the limit as $n\to\infty$ of the rank of G_n.
I. Halperin (Kingston, Ont.).

Wavre, Rolin. Sur les valeurs propres des opérateurs hermitiens. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 61–63 (1942). [MF 14200]

Wavre, Rolin. La décomposition spectrale des opérateurs hermitiens. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 112–115 (1942). [MF 14201]

Wavre, Rolin. Sur les équations linéaires à opérateurs hermitiens. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 157–159 (1942). [MF 14206]

Vigier, Jean-Pierre. Quelques résultats complémentaires à la théorie de l'itération des opérateurs de M. Wavre. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 159–162 (1942). [MF 14207]

Wavre, Rolin. Remarques à propos de l'itération des opérateurs hermitiens. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 229–233 (1942). [MF 14210]

Wavre, Rolin. Le spectre et la théorie du rang. C. R. Séances Soc. Phys. Hist. Nat. Genève **60**, 194–196 (1943). [MF 14217]

Wavre, Rolin. Sur la décomposition spectrale des opérateurs hermitiens. C. R. Séances Soc. Phys. Hist. Nat. Genève **61**, 154–155 (1944). [MF 14222]

These notes present a theory of the spectral resolution of the square of a bounded Hermitian operator. The development is based on a process of iterating the operator and the use of Kellogg's inequality and is interesting both for its simplicity and historical connections. *F. J. Murray.*

Kakutani, Shizuo. Topological properties of the unit sphere of a Hilbert space. Proc. Imp. Acad. Tokyo **19**, 269–271 (1943). [MF 14822]

Let K be the set of all elements x in Hilbert space such that $\|x\|\leqq 1$ and let S be the set of all x in K for which $\|x\|=1$. The author exhibits a homeomorphism (in the norm topology) of K onto itself which has no fixed point. Using this homeomorphism he constructs a continuous mapping of K onto S which leaves the elements of S fixed and shows that for each x_0 in S the identity mapping of S onto itself is homotopic to the mapping which carries each element of S into x_0. *G. W. Mackey* (Cambridge, Mass.).

Itô, Kiyosi. A screw line in Hilbert space and its application to the probability theory. Proc. Imp. Acad. Tokyo **20**, 203–209 (1944). [MF 14884]

"A. Kolmogoroff has investigated the spectralization of the screw line in Hilbert space [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 6–9 (1940); these Rev. **2**, 220] . . . In this note I will show his results, although the proofs may run in the same way as his own. And I will apply the results to the theory of two-dimensional Brownian motions." The author's proofs run essentially along the lines given by von Neumann and Schoenberg [Trans. Amer. Math. Soc. **50**, 226–251 (1941); these Rev. **3**, 37], which the author also quotes in a footnote. The author's applications to Brownian motion were already discussed by Kolmogoroff in his paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 115–118 (1940); these Rev. **2**, 220], which the author does not quote. *I. J. Schoenberg.*

Yosida, Kôsaku, and Nakayama, Tadasi. On the semi-ordered ring and its application to the spectral theorem. Proc. Imp. Acad. Tokyo **18**, 555–560 (1942). [MF 14781]

Yosida, Kôsaku, and Nakayama, Tadasi. On the semi-ordered ring and its application to the spectral theorem. II. Proc. Imp. Acad. Tokyo **19**, 144–147 (1943). [MF 14806]

The second paper corrects and continues the first. This work is very closely related to two papers of M. H. Stone [Proc. Nat. Acad. Sci. U. S. A. **26**, 280–283 (1940); **27**, 83–87 (1941); these Rev. **1**, 338; **2**, 318] to which no reference is made and to a paper of Vernikoff, Krein and Tovbin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 785–787 (1941); these Rev. **2**, 314] to which reference is made. A representation theorem is proved for partially ordered Abelian groups G with partially ordered Abelian groups of operators. This yields several representation theorems for partially ordered rings R which in turn give a general spectral theorem and the theorem of M. and S. Krein and Kakutani on Banach lattices [M. and S. Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 427–430 (1940); S. Kakutani, Ann. of Math. (2) **42**, 994–1024 (1941); these Rev. **2**, 222; **3**, 205]. It also yields a theorem on the additive group of bounded automorphisms of G.

First representation theorem: G is operator isomorphic to an additive group of bounded real functions over the space of maximal allowable normal subgroups (allowable: operator invariant; normal: contains, with any y, all x for which $0\leqq x\leqq y$). Second representation theorem: if (1) R (not necessarily assumed Abelian or associative) admits real multipliers, (2) for any $x\epsilon R$ there is a real number $\alpha=\alpha(x)$ such that $-\alpha e\leqq x\leqq \alpha e$, e being the identity of R, (3) $\inf \epsilon e=0$ (inf for all real numbers $\epsilon>0$), then R is isomorphic to a ring of bounded real functions on a compact space. Two variations of this theorem are obtained, one stating that if R is a Banach space under the norm $\|x\|=\inf \alpha$ for $-\alpha e\leqq x\leqq \alpha e$ then R is isomorphic to the ring of all continuous real functions on the compact space. This yields the Krein-Kakutani theorem referred to above and, with the additional assumption that increasing sequences in R have order limits, the general spectral theorem.

The procedure for getting the spectral resolution of $x\epsilon R$ is the same as that mentioned in the first reference to M. H. Stone given above. The spectral resolution of x is a family of idempotents e_λ in R with the usual properties. It is obtained by considering x as a function $x(M)$ on the maximal ideals, taking the set of points M at which $x(M)\leqq\lambda$, and proving that the characteristic function of this set differs on a set of first category from a unique element, denoted by e_λ, of R. These e_λ comprise the desired spectral family. *W. Ambrose* (Ann Arbor, Mich.).

Yosida, Kôsaku. Normed rings and spectral theorems. Proc. Imp. Acad. Tokyo **19**, 356–359 (1943). [MF 14831]

A proof via normed rings of the existence of a simultaneous spectral resolution of an Abelian family A of normal operators on Hilbert space. Let A' be the set of bounded linear operators that commute with A and $A''=(A')'$. Then A'' is a normed ring in the sense of Gelfand with $\|x\|$ the bound of the operator x. First, using standard properties of Hilbert space operators and Gelfand's theorems [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51] it is shown that A'' is isomorphic to the ring of all continuous functions on a compact space, with adjoint operators becoming complex-conjugate functions. [This is also a consequence of a general theorem of Gelfand and Neumark, Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 197–213 (1943); these Rev. **5**, 147.] Then the proof is completed as in the paper reviewed above. [It should be noted that Stone [Proc. Nat. Acad. Sci. U. S. A. **26**, 280–283 (1940); these Rev. **1**, 338] mentions this general method of proof, for an Abelian ring of self-adjoint operators, as an application of his results that such a ring is a ring of all continuous functions. *W. Ambrose* (Ann Arbor, Mich.).

Yosida, Kôsaku. Normed rings and spectral theorems. II. Proc. Imp. Acad. Tokyo **19**, 466–470 (1943). [MF 14844]

[Cf. the preceding review.] A normed ring treatment of the theory of completely continuous operators on a Banach space [S. Banach, Théorie des Opérations Linéaires, Warsaw, 1932, pp. 151–164] including the Nikolski extension [C. R. (Doklady) Acad. Sci. URSS (N.S.) **11**, 315–319 (1926)] in which only some power is assumed completely continuous. Let V be completely continuous, I the identity operator, and complete the ring generated by V and I in the uniform topology, obtaining a normed ring R with the bound of an operator for its norm. First it is shown without using R that the spectrum of V is exclusively point spectrum, hence complete continuity ensures that the spectrum is countable with 0 the only possible limit point. Now R enters. Consider the natural homomorphism of R into the ring of complex-valued continuous functions on its space of maximal ideals and denote the image of $T\epsilon R$ by $T(M)$ (M any maximal ideal). From known theorems [Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51] the spectrum of V is the range of $V(M)$. Hence, using the above, the space of maximal ideals is countable, $M_1, \cdots, M_n, \cdots$ and $V(M_i)=\lambda_i$, the λ_i's being the spectrum of V.

Corresponding to each λ_i an idempotent I_i is found in R and a consideration of the maximal ideals in RI_i and $R(I-I_i)$ shows that the underlying Banach space B can be split into two subspaces, on one of which λ_i-V has an inverse and on the other of which it is nilpotent. The operator V^* dual to V generates a ring R^* in the same way that V generated R and there is an obvious natural isomorphism between R and R^*. Hence all facts about the above I_i are equally true about corresponding idempotents I_i^* in R^*. This implies immediately that V and V^* have the same spectra and, with a slight argument, proves the multiplicities are also the same. *W. Ambrose.*

Yosida, Kôsaku. Normed rings and spectral theorems. III. Proc. Imp. Acad. Tokyo **20**, 71–73 (1944). [MF 14874]

An abstract extension of some results of Temple and Collatz [Proc. London Math. Soc. (2) **29**, 257–280 (1929); Math. Z. **46**, 692–708 (1940); these Rev. **2**, 312] concerning the greatest eigenvalue of an integral equation. Let R be the normed ring of real continuous functions $f(x)$ on a compact space C, and L a continuous positive linear functional on R so that there exists a regular measure m on C for which Lf is the integral of f with respect to m. A number μ, called the "largest spectral value of L with respect to f," is defined as follows: let $\varphi(\lambda)$ be the distribution function of f for this measure m, that is, $\varphi(\lambda)=m[x|f(x)<\lambda]$; then μ is defined as the largest λ which is a point of increase of φ. The point of the paper is to prove certain inequalities and limiting relations, for $f>0$, between Lf^n and μ; we state one

of these. Let $Lf^{2n+1}/Lf^{2(n+1)}=1/\mu_n$ and $(Lf^{2n}/Lf^{2(n+1)})^{\frac{1}{2}}=1/\nu_n$. Then $\lim 1/\mu_n=\lim 1/\nu_n=1/\mu$.

This theorem applies to any self adjoint operator T on Hilbert space because T is contained in a ring of operators isomorphic to such an R [Gelfand and Neumark, Rec. Math. [Mat. Sbornik] N.S. **12**(54), 197–213 (1943); these Rev. **5**, 147; see also the preceding review]. In terms of the linear functionals $L(S)=(Sy, y)$, S in the ring containing T, y any fixed point of Hilbert space, the definition gives the ordinary largest spectral value. The apparently special case of a ring of operators on Hilbert space is as general as the above R by the Gelfand-Neumark theorem that every normed $*$-ring is a ring of operators on Hilbert space.

W. Ambrose (Ann Arbor, Mich.).

Yosida, Kôsaku. Normed rings and spectral theorems. IV. Proc. Imp. Acad. Tokyo **20**, 183–185 (1944). [MF 14881]

A continuation of the preceding paper, proving similar facts about the second largest spectral value, etc. In the notation of the preceding review, it is assumed that μ is an isolated jump of φ, and the next largest λ at which φ increases is denoted by μ'. Defining, in terms of the given Lf, a new positive linear functional L' by $L'g=L(\mu g-fg)/L(\mu-f)$, μ' is the largest spectral value of L' with respect to f, so applying the previous theorem to L' gives corresponding results about μ'. Some relations between the original μ_j, ν_j and μ' are obtained, not involving μ, and these are assigned the name "practical formulae."

W. Ambrose (Ann Arbor, Mich.).

Yosida, Kôsaku. Normed rings and spectral theorems. V. Proc. Imp. Acad. Tokyo **20**, 269–273 (1944). [MF 14892]

Yosida, Kôsaku. Normed rings and spectral theorems. VI. Proc. Imp. Acad. Tokyo **20**, 580–583 (1944). [MF 14928]

[Cf. the preceding review.] The second paper continues and corrects the first. They contain proofs of the Krein-Weil generalization of the Plancherel theorem to locally compact Abelian groups [M. Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 484–488 (1941); A. Weil, L'integration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940; these Rev. **2**, 316; **3**, 198] and the Powzner-Raikov-Weil extension of the Herglotz-Bochner theorem on the representation of positive definite functions, the second being made to follow from the first. The proof of the generalized Plancherel theorem is similar to Krein's proof but the Fourier transform is considered for all $z \in R(G)$, where $R(G)$ is the ring of integrable functions on G with a unit adjoined. The proof is made to depend on theorems about rings of operators on Hilbert space by considering for each $z \in R(G)$, $z=\lambda e+x$, $x \in L_1(G)$, the operator defined for all $y \in L_2(G)$ by $T_z y=\lambda y+x^*y$ (x^*y the convolution of x and y) and considering the closure, in the uniform topology, of the operators T_z.

W. Ambrose.

Lewitan, B. M. A generalization of the operation of translation and infinite hypercomplex systems. I. Rec. Math. [Mat. Sbornik] N.S. **16**(58), 259–280 (1945). (English. Russian summary) [MF 14582]

The author generalizes the notion of translation by assuming the existence of families T^s of bounded linear transformations on function or sequence spaces, with the formal properties of the ordinary translation operator. For example, if the space is L^2 let $T_t^s f(t)$ denote the result of operating on f with the operator T^s, s being a parameter; it is postulated that (I) for fixed t the function $T_t^s f(t)$ belongs to L^2 as a function of s; (II) $T_s^r T_t^s f(t)=T_t^s T_t^r f(t)$; (III) $T_t^s U_t^s=U_t^s T_t^s$; (IV) $U_s^r T_t^s=T_t^s U_t^r$; here U^s denotes the adjoint of T^s. It is then possible to define the almost-periodicity of $f(t)$ with respect to the family of transformations, and to obtain conditions on T^s which guarantee that the "right" functions have the property of almost periodicity.

This program is carried through in some detail for the space of functions of integrable square on an n-dimensional Euclidean space, where the transformations T^s are of kernel type. In the case of sequence spaces the operators are defined by cubic matrices, concerning which more is said in the sequel to this paper, reviewed below.

H. Pollard.

Lewitan, B. M. A generalization of the operation of translation and infinite hypercomplex systems. II. Rec. Math. [Mat. Sbornik] N.S. **17**(59), 9–44 (1945). (English. Russian summary) [MF 14591]

Let l^2 be the space of all complex sequences $\{x_r\}$ such that $\sum x_r^2$ converges. Define $T_q^p=\sum_{r=1}^{\infty} a_{pqr}x_r$, where the coefficients are chosen so that the conditions for an operator of generalized translation are satisfied [see the preceding review]. In the present paper these coefficients are defined by expressions of the form

$$a_{pqr}=\int_a^b \omega_p(x)\omega_q(x)\omega_r(x)p(x)dx,$$

where either (i) the ω_i are the orthonormal polynomials associated with a suitably restricted weight function $p(x)$ or (ii) the ω_i are the eigenfunctions of a second-order Sturm-Liouville problem and $p(x)\equiv 1$. In these cases the domain of the operators can be extended to all bounded sequences $\{x_r\}$. For such sequences the concept of almost-periodicity with respect to the family of operators can be defined, and analogues of the approximation theorem and Parseval's equality established. The relevant formulas are too cumbersome to reproduce in a review.

H. Pollard.

Lewitan, B. The duality law for the generalized operation of translation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 387–389 (1945). [MF 14424]

D. Raikov has shown [same C. R. (N.S.) **30**, 589–591 (1941); these Rev. **2**, 317] that the Pontrjagin duality law, for locally compact groups satisfying the second countability axiom, follows from Plancherel's theorem. The author shows that a generalization of this result can be obtained by using his results for generalized translations [same C. R. (N.S.) **47**, 3–6, 159–161, 318–321 (1945); these Rev. **7**, 126, 127]. For every operation of generalized translation there exists an operation of generalized translation "dual" to it and, under certain additional conditions (always satisfied in the group case), the duality law holds.

C. E. Rickart (New Haven, Conn.).

Godement, Roger. Sur les fonctions de type positif. C. R. Acad. Sci. Paris **221**, 69–71 (1945). [MF 14238]

Let C be the Banach space of bounded, complex-valued continuous functions on a locally compact group G. A function $f \in C$ is said to be of "positive type" on G provided

$$\sum_{i,j=1}^{n} \rho_i\bar{\rho}_j f(s_i^{-1}s_j)\geqq 0,$$

for arbitrary numbers $\rho_1, \cdots, \rho_n$ and elements $s_1, \cdots, s_n \in G$. The class of all such functions is denoted by $\mathfrak{T}$, its linear extension by $\mathfrak{U}$ and the closure of $\mathfrak{U}$ in C by $\mathfrak{P}$. If H is a

Hilbert space and U_x is a continuous unitary representation of G in H, then the function $f(x)=(\varphi, U_x\varphi)$, for each $\varphi\epsilon H$, is an element of $\mathfrak{T}$. Conversely, every element $f\epsilon\mathfrak{T}$ determines a unitary representation U_x of G such that $f(x)=(\varphi, U_x\varphi)$. Every $f\epsilon\mathfrak{L}$ is uniformly continuous on G. The manifold $\mathfrak{L}$ is invariant under translation. The smallest closed convex subset of C which is invariant (left and right) and which contains an element of $\mathfrak{L}$ is weakly compact in C.

C. E. Rickart (New Haven, Conn.).

Godement, Roger. Sur les propriétés ergodiques des fonctions de type positif. C. R. Acad. Sci. Paris **221**, 134–136 (1945). [MF 14245]

An element $f\epsilon C$ [see the preceding review for notations] is said to be "left (right) ergodic" provided the closed convex set determined by left (right) translations of f contains a constant. If f is both left and right ergodic it is said to be ergodic. If f is ergodic, the common value of its left and right means [von Neumann, Trans. Amer. Math. Soc. **36**, 445–492 (1934)] is denoted by $M(f)=M_t[f(t)]$. By application of results obtained in the note reviewed above and a theorem of G. Birkhoff [Proc. Nat. Acad. Sci. U. S. A. **25**, 625–627 (1939); these Rev. **1**, 148], it can be proved that every $f\epsilon\mathfrak{L}$ is ergodic. If one defines $f\times g(x)=M_t[f(xt^{-1})g(t)]$, then $S_f(x)=f\times\tilde{f}(x)$, where $\tilde{f}(x)=f(x^{-1})$, is called the spectral function of f. If $f, g\epsilon\mathfrak{L}$, then $f\times g\epsilon\mathfrak{U}$ and $S_f\epsilon\mathfrak{T}$. If $f\epsilon\mathfrak{L}$, then S_f can be represented uniquely in the form $S_f(x)=\sum\lambda_k\Phi_k(x)+\varphi(x)$, where $\varphi\epsilon\mathfrak{T}$, $S_\varphi=0$, $\lambda_k\geqq\lambda_{k+1}\geqq 0$, $\Phi_k\epsilon\mathfrak{T}$, $\Phi_k\times\Phi_l=\Phi_k$ or 0 according as $k=l$ or $k\neq l$, and the series converges uniformly. If $\Phi\epsilon\mathfrak{T}$ and $\Phi\times\Phi=\Phi$, then Φ is almost periodic and determines a finite dimensional unitary representation of G. *C. E. Rickart* (New Haven, Conn.).

Riesz, Frédéric. Sur la théorie ergodique. Comment. Math. Helv. **17**, 221–239 (1945).

Under this title are presented two addresses given by the author at the University of Geneva. The first is based on the ergodic theorem of G. D. Birkhoff; the second, on that of von Neumann. The author introduces a new short proof of the so-called maximal ergodic theorem on which Birkhoff's result is based. The second lecture discusses methods developed principally by G. Birkhoff, the author, and the reviewer. *E. R. Lorch* (New York, N. Y.).

Niemytzki, V. Systèmes dynamiques sur une multiplicité intégrale limite. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 535–538 (1945). [MF 14417]

Let X be a complete metric space in which a continuous flow $f^t(-\infty<t<+\infty)$ is defined. The parameter t will be referred to as the time. The motion determined by the point x of X (that is, the set $f^t(x)$, $-\infty<t<+\infty$) is positively (negatively) stable [in the sense of Lagrange] if the positive (negative) semi-motion of x is conditionally compact. Let L be a positive (negative) semi-motion of a motion M which is positively (negatively) stable and let K be the set of ω (α) limit points of L. Furthermore, it is supposed that K does not contain L. Then L is said to be uniformly asymptotic to K if, corresponding to $\epsilon>0$, there exists a positive real number T such the segment of L determined by any time interval T comes within ϵ of every point of K. It is shown that a necessary and sufficient condition that K be a minimal set is that L be uniformly asymptotic to K.

The positive semi-motion $f^t(x)$, $t>0$, is said to have property S^+ of Franklin if, corresponding to $\epsilon>0$, there exists an $\eta>0$ such that $\rho[f^{t_1}(x), f^{t_2}(x)]<\eta$, $t_1>0$, $t_2>0$, implies $\rho[f^{t_1+t}(x), f^{t_2+t}(x)]<\epsilon$ for all positive t. If L has property S^+ of Franklin, it follows readily that the set of transformations f^t is equi-uniformly continuous and thus the motions of K are almost periodic. *G. A. Hedlund.*

Kawada, Yukiyosi. Über die masstreuen Abbildungen vom Mischungstypus im weiteren Sinne. Proc. Imp. Acad. Tokyo **19**, 520–524 (1943). [MF 14849]

Consider two sets Ω and Ω' with measures m and m' and $m(\Omega)=m'(\Omega')=1$. Let T (T') be a measure preserving transformation of Ω onto Ω (Ω' onto Ω') and consider the transformation $\bar{T}=T\times T'$ on $\Omega\times\Omega'$ which preserves the product measure $m\times m'$. Two theorems are proved, each giving conditions for T to be of mixing type. The first asserts that this is the case if and only if $\bar{T}$ is ergodic for every ergodic choice of T'. The second is closely related to the mixing theorem. *W. Ambrose* (Ann Arbor, Mich.).

Kawada, Yukiyosi. Über die masstreuen Abbildungen in Produkträumen. Proc. Imp. Acad. Tokyo **19**, 525–527 (1943). [MF 14850]

Using the notation of the preceding review, let G be the point spectrum of T; G consists of all real numbers λ for which there is an $f(\omega)\epsilon L_2(\Omega)$, $f\neq 0$, such that $f(T\omega)=e^{i\lambda}f(\omega)$. Let G' be the point spectrum of T'. Two theorems are proved: (1) $\bar{G}$, the point spectrum of $\bar{T}=T\times T'$, is the set of all $\lambda+\lambda'$, where $\lambda\epsilon G$, $\lambda'\epsilon G'$; (2) $\bar{T}$ is ergodic if and only if both (i) T and T' are ergodic, (ii) $G\cap G'=(0)$. One corollary of these is the first theorem in the paper reviewed above. *W. Ambrose* (Ann Arbor, Mich.).

Kakutani, Shizuo. Induced measure preserving transformations. Proc. Imp. Acad. Tokyo **19**, 635–641 (1943). [MF 14862]

A measure preserving transformation (m.p.t.) $\varphi(\omega)$ of a measure space Ω (sum of countably many sets of finite measure) is said to be isomorphic to a m.p.t. $\varphi'(\omega')$ of a measure space Ω' if there exists a m.p.t. χ of Ω onto Ω' such that $\chi(\varphi(\omega))=\varphi'(\chi(\omega))$ almost everywhere. If φ is an ergodic m.p.t. of Ω, and $\Omega'\subset\Omega$ is any subset with $m\Omega'>0$, it is shown that φ induces an ergodic m.p.t. $\varphi_{\Omega'}$ of the relative measure space Ω' onto itself, defined almost everywhere by putting $\varphi_{\Omega'}(\omega)=\varphi^n(\omega)$, where $n=n(\omega)$ is the least positive integer such that $\varphi^n(\omega)\epsilon\Omega'$. Any m.p.t. ψ isomorphic to a transformation $\varphi_{\Omega'}$ induced by φ is called a derivative of φ, and φ is called a primitive of ψ. This relation, denoted by $\varphi\vdash\psi$, is transitive, but $\varphi\vdash\psi$ and $\psi\vdash\varphi$ do not in general imply that φ and ψ are isomorphic. A result attributed to von Neumann is that two ergodic m.p.t. φ and ψ have a derivative in common if and only if they have a primitive in common, and in this case φ and ψ are called equivalent. The principal result announced in the present note is that two ergodic m.p.t. φ and φ' of complete measure spaces are equivalent if and only if each "flow built on φ under a function f" (in the sense of Ambrose [Ann. of Math. (2) **42**, 723–739 (1941); these Rev. **3**, 52]) is isomorphic to a flow built on φ' under a function f', and each flow built on φ' is isomorphic to a flow built on φ. The author conjectures that any two ergodic m.p.t. are equivalent.

J. C. Oxtoby (Bryn Mawr, Pa.).

Terlezki, J. P. Concerning the justification of replacement of time averages by phase averages in statistical mechanics. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 543–545 (1945). [MF 14420]

The author states that since the "ergodene hypothesis" is not only known to be unprovable under general assumptions, but is actually false for a large class of dynamical

systems, for example, for conditionally periodic ones, some other justification of Gibbs's method of phase average must be found. [Evidently he is not acquainted with von Neumann's mean ergodic theorem or with G. D. Birkhoff's ergodic theorem, and is referring to the older formulations of the "quasi-ergodic hypothesis."] He holds that such a justification is shown by "a simple physical analysis of the phenomena and experiments to which statistical mechanics is applicable" not to require the exact equality of time and phase averages, but only their approximate equality for most initial points of phase space. "What can and must be used in the proof is that all systems, to which methods of statistical mechanics are applied, possess a large number of degrees of freedom." The theorem needed is regarded as equivalent to the relation

$$\overline{(\bar{F}^t-\bar{F})^2}\to 0, \qquad N\to\infty, \tag{1}$$

similar to one considered by Khintchine [Matematičeskie Osnovaniya Statističeskoi Mehaniki [Mathematical Principles of Statistical Mechanics], OGIZ, Moscow-Leningrad, 1943, p. 54]. Here $F=F(X)$, X is a point of phase-space, N is the number of degrees of freedom, simple bars denote phase averages and bars followed by superscript t denote time averages.

The proof of (1) is given under the following assumptions. (a) The system is an isolated linear system of N connected linear oscillators (coordinates and momenta denoted by x with subscripts); (b) the integrals of the system are pairwise statistically independent (with respect to phase averages); (c) $F(X)=x_k(t)x_l(t+\tau)$, τ being a fixed time interval. Generalization to an arbitrary system performing conditionally periodic motion is stated to be possible.

B. O. Koopman (New York, N. Y.).

GEOMETRY

Coolidge, J. L. Two dimensions. Amer. Math. Monthly **52**, 557–562 (1945). [MF 14486]

An article designed to refute the dictum: "In geometry the only interesting theorems are those which are true in any number of dimensions." In support of the argument, examples are given to illustrate the richness in content of two-dimensional geometry as compared with the "higher" geometries. *W. V. D. Hodge* (Cambridge, England).

***Efimov, N. V. Vysšaya Geometriya. [Higher Geometry].** OGIZ, Moscow-Leningrad, 1945. 487 pp. (Russian)

This university textbook treats mainly the foundations of geometry. The following summary of the chapters gives an idea of the subjects treated. (I) Short historical introduction. (II) Hilbert's axioms for Euclidean geometry and their most important consequences. (III) Lobachevskian geometry: development from the axioms by elementary methods; Poincaré's model. (IV) Further examination of Hilbert's axioms: freedom from contradiction, proved by the arithmetical model; independence of the principal axioms; completeness, taken in the sense of isomorphy of models. (V) Direct deduction of the quadratic form which defines the line-element in Lobachevskian geometry; realization of this geometry by domains on surfaces of constant curvature; introduction to abstract differential geometry; the space-forms of Cayley-Klein. (VI) Axiomatic foundation of projective geometry; introduction of coordinates; projective transformations; generation of conics by projective pencils. (VII) The Erlanger Programm. Projective foundation of non-Euclidean geometry. The book is written in a clear style and meets a high standard of rigor.

A. Heyting (Laren).

Hjelmslev, Johannes. The Pythagorean theorem. Mat. Tidsskr. B. **1945**, 49–57 (1945). (Danish) [MF 14260]

It is shown that Euclidean geometry is to a great extent independent of the assumption that a segment determines uniquely the line which contains it. The axioms are roughly the following. The segment connecting two points is unique. For every line there is a motion (reflection) that leaves all points of the line, but no other point of the plane, fixed. For a given point p and a given line L there is exactly one line through p perpendicular to L. Each segment has a center. (All these terms are defined in terms of reflections.) For two given rays r_1, r_2 with the same origin there is a reflection carrying r_1 into r_2. If in a quadrangle three angles are right angles then the fourth angle is right. The parallel and Archimedean axioms are not assumed. These axioms suffice to prove the theorem of Pythagoras and are consistent with the further axiom that there is a segment that belongs to two different lines. *H. Busemann.*

Rossier, Paul. Sur la géométrie des sphères et des cercles et la définition du plan et de la droite. C. R. Séances Soc. Phys. Hist. Nat. Genève **60**, 284–286 (1943). [MF 14220]

The author gives a brief analytical discussion of the sphere as an undefined element of axiomatic elementary geometry of three dimensions. The sphere has the psychological advantage that we can have a material model of it, while we cannot have such a model of the plane due to its infinite extent. Other advantages, as well as disadvantages, are pointed out. *N. A. Court* (Norman, Okla.).

Rossier, Paul. Sur la géométrie du compas à pointes sèches et celle de l'empan. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 47–48 (1942). [MF 14197]

The author states that in all construction problems solvable by means of straight edge and compass (second degree problems) the compass may be replaced by a pair of dividers (compas à pointes sèches) with fixed opening.

L. M. Blumenthal (Columbia, Mo.).

Fejes, László. Einige Bemerkungen über die dichteste Lagerung inkongruenter Kreise. Comment. Math. Helv. **17**, 256–261 (1945).

The density D of a system of circular discs $\{k_i\}$ (of areas k_i) which cover the plane is defined by considering an arbitrary sequence of similar convex regions T_n, of areas T_n tending to ∞. If $F(T_n)$ is the area common to T_n and the system $\{k_i\}$, then $D=\lim F(T_n)/T_n$. Two problems are of interest: the maximum density D_1 for nonoverlapping discs and the minimum density D_2 for overlapping discs which completely cover the plane. For congruent circular discs the exact values are $D_1=1/\phi(6)\cong 0.907$, $D_2=1/\psi(6)\cong 1.209$, where $\phi(\nu)=\tan(\pi/\nu)/(\pi/\nu)$, $\psi(\nu)=\sin(2\pi/\nu)/(2\pi/\nu)$; and these values are obtained when the centers of the circles form a regular hexagonal pattern. If the relative sizes of circles are unrestricted, D_1 and D_2 both become 1, but

if the ratios of the areas of any two circles in the system $\{k_i\}$ lie between δ and $1/\delta$, where $0<\delta<1$, then it is shown that $D_1 \leqq 1/\phi(6/\delta)$ and $D_2 \geqq 1/\psi(6/\delta)$. If circles of only two sizes k and K are used and if the average numbers of vertices of polygons associated with the two sizes are $6-\alpha$ and $6+\alpha$, respectively, then we have $(k+K)/D_1 \geqq k\phi(6-\alpha)+K\phi(6+\alpha)$, $(k+K)/D_2 \leqq k\psi(6-\alpha)+K\psi(6+\alpha)$. The density $D_0=\pi(1-2^{-\frac{1}{2}})$ [misprinted in the paper] is assumed by the regular square and octagon pattern ($\alpha=2$). This value $D_0=0.92015$ gives a lower bound for D_1, and it is presumed but not proved that this cannot be exceeded when nonoverlapping discs of two sizes are considered.
J. S. Frame (East Lansing, Mich.).

Alaoglu, Leonidas, and Giese, J. H. Uniform isohedral tori. Amer. Math. Monthly **53**, 14–17 (1946). [MF 15306]

These polyhedra, in ordinary space, are topologically regular, of genus 1 (having six triangles or four quadrangles at each vertex), and are metrically isohedral in the local sense, that is, the faces are congruent but not all surrounded alike by other faces. To construct one of them, take a convex isohedral octahedron or hexahedron having two opposite faces which are not parallel, and adjust the measurements so as to make the dihedral angle between these opposite faces π/n. Reflections in the planes of these faces will generate a ring of $2n$ such octahedra or hexahedra, whose exposed faces (six or four for each) form the desired torus. Thus the faces consist of $12n$ scalene triangles ($n \geqq 3$) or $8n$ irregular quadrangles ($n \geqq 4$). In the former case a variant of the method leads to a generalization which suggests that it might be possible to derive a knotted torus (for a sufficiently large value of n). It is not yet known whether a uniform isohedral torus can have fewer than 36 triangular or 32 quadrangular faces, nor whether one can have hexagonal faces (three at each vertex). *H. S. M. Coxeter.*

Fenchel, W. An elementary mechanical realization of Levi-Civita's infinitesimal parallel-displacement on surfaces. Mat. Tidsskr. B. **1945**, 110–112 (1945). (Danish) [MF 14265]

Let a circular disk D be fastened to an axis a perpendicular to D at its center p in such a way that D can rotate without friction but not slide. Let the resultant moment about a of the exterior forces acting on D be zero. It is assumed that the mass of D has rotational symmetry. On D draw a vector v with origin p. If the system starts at rest and is moved so that p traverses a curve k on a surface S and a constantly coincides with the normal of S, then v moves as under parallel displacement along k. It follows, in particular, that on a sphere this system can be used as a planimeter. *H. Busemann* (Northampton, Mass.).

Wundheiler, Alexander W. On the MacAdam ellipses. J. Opt. Soc. Amer. **35**, 767–771 (1945). [MF 14540]

Given two homothetic ellipses of major and minor axes $a(1-p)$, $b(1-p)$ and $a(1+p)$, $b(1+p)$, respectively, consider all other ellipses which are concentric with them and lie in the strip bounded by them. If θ is the angle between the major axis of the given ellipse and that of one of the other ellipses, the maximum value of θ is given by $\sin 2\theta = 4p/(1-p^2)ef$, where e is the eccentricity of the ellipse, called the "average" ellipse, whose axes are a, b, respectively, and $f=e/(1-e^2)^{\frac{1}{2}}$. For a given θ, the extreme values of the ratio x of the square of the major axis of the other ellipse to that of the "average" ellipse are the roots of the equation $(1+f^2\beta^2)^2x-2(1+p^2)x+(1-p^2)^2(1-e^2\beta^2)=0$, where $\beta=\sin\theta$. A similar relation holds for the minor axis.

The results are applied in considering the validity of hypotheses made by Moon and Spencer [same J. 33, 260–269 (1943)] concerning the orientation of ellipses of standard deviation of color matching given by MacAdam [same J. 32, 247–274 (1942)]. The author quotes from a communication by MacAdam to the effect that the hypotheses require greater deviations from the data than are explainable by experimental error. *A. J. Kavanagh.*

Thébault, Victor. Sur la géométrie du tétraèdre. C. R. Acad. Sci. Paris **218**, 820–822 (1944). [MF 15329]

Fettis, H. E. The Fermat and Hessian points of a triangle. Amer. Math. Monthly **53**, 74–78 (1946). [MF 15463]

Claeys, A. Sur la courbe cappa. Mathesis **54**, 230–236 (1940). [MF 15523]

La courbe cappa ou quartique de Gutschoven est le lieu du point M tel que si O est un point fixe sur une droite d, et si la perpendiculaire élevée en M sur OM coupe d en N, le segment MN soit constant; autrement: M est le point de contact de OM avec le cercle (N, NM) ou (N, a), N se déplaçant sur d. *Extract from the paper.*

Backes, F. Sur les droites de Simson. Mathesis **54**, 301–304 (1942). [MF 15528]

Goormaghtigh, R. Sur une enveloppe. Mathesis **54**, 295–298 (1942). [MF 15526]

La recherche de l'enveloppe de la droite de Simson d'un point fixe d'une circonférence par rapport à un triangle inscrit, de forme donnée, tournant autour du centre de la circonférence, donne lieu à des résultats intéressants.
From the author's summary.

Goormaghtigh, R. Extension au polygone inscriptible du théorème sur l'enveloppe des droites de Simson d'un triangle. Mathesis **54**, 362–365 (1943). [MF 15532]

Goormaghtigh, R. Sur deux transformations géométriques. Mathesis **54**, 222–227 (1940). [MF 15522]

M. H. Lorent a considéré deux transformations géométriques: la première (T) fait correspondre à une courbe plane le lieu des extrémités de ses sous-tangentes polaires; la seconde (T_1) fait correspondre à une courbe plane celle décrite par l'intersection de la tangente en un point variable avec la droite menée par un point fixe et ayant une direction symétrique, par rapport à une direction fixe, de celle du rayon vecteur mené de ce point fixe au point variable de la courbe. Dans les considérations qui suivent, nous appliquons à l'étude de la première deux méthodes qui conduisent à des propriétés générales et à de nombreux cas particuliers; quant à la seconde transformation, nous montrons qu'elle se déduit de la première par une affinité complexe. *From the author's summary.*

Godeaux, Lucien. Sur une propriété des quadriques. Revista Mat. Hisp.-Amer. (4) **5**, 187–190 (1945). [MF 15206]

Étant donnés une quadrique non conique Q, deux points M, N de cette quadrique tels que la droite MN n'appartienne pas à la quadrique, α, β deux plans conjugués passant par MN, a une tangente à Q en M située dans α, b une tangente à Q en N située dans β, le lieu des tangentes à Q

s'appuyant sur a et b est formé de deux hyperboloïdes H_1, H_2, respectivement circonscrits à Q le long de deux coniques φ, ψ situées dans des plans passant par M, N, conjugués par rapport à Q et partageant harmoniquement les plans α, β. Les génératrices des deux modes des hyperboloïdes H_1, H_2 touchent Q le long de φ, ψ respectivement et s'appuyent sur les conjuguées a' de a et b' de b.

Extract from the paper.

Araujo, Roberto. Some observations on the theory of the conics. Revista Mat. Hisp.-Amer. (4) 5, 105–110 (1945). (Spanish) [MF 15201]

Mandan, Ram. Distance in homogeneous co-ordinates. Proc. Lahore Philos. Soc. 7, no. 1, 54–55 (1945). [MF 15177]

Deaux, R. Conique circonscrite à un triangle réel. Mathesis 54, 237–242 (1940). [MF 15524]

Deaux, R. Sur l'involution quadratique. Mathesis 54, 357–362 (1943). [MF 15531]

Deaux, R. Sur les involutions harmoniques à une homographie binaire. Mathesis 54, 315–317 (1942). [MF 15530]

Motzkin, Th. A 5 curve theorem generalizing the theorem of Carnot. Bull. Amer. Math. Soc. 51, 972–975 (1945). [MF 14471]

If a, b, c, p, q are any five coplanar algebraic curves, and $a(x)=0$, etc., denote their respective equations in homogeneous coordinates, let us indicate by abp the resultant of the three forms $a(x)$, $b(x)$, $p(x)$, that is, $abp=\prod p(x)$, where the product in the right-hand side is extended over all the points x of intersection of a and b. The author defines the cross-ratio $a(bp/cq)$ determined on the curve a by the curves b, c, p, q taken in this order as $a(bp/cq)=(abp:abq):(acp:acq)$ and remarks that $a(bp/cq)\cdot b(cp/aq)\cdot c(ap/bq)=\pm 1$, as all factors cancel out. This result includes Menelaos' theorem and its known extensions, due to Newton and Carnot, as very special cases, and is only one of a number of kindred theorems. [For further extensions, cf. the following review.] Dualizing and replacing two of the curves by the circular points at infinity, the author obtains that the sum of the angles of the asymptotes of a curve with a given line is determined by the foci of the curve, and that the sum of the angles formed by the common tangents of two curves with a given line is determined by the two sets of foci of the two curves. [The last theorem is ascribed by the author to Laguerre, but the reviewer has been able to find only the particular case when one of the two curves is of class one in E. Laguerre, C. R. Acad. Sci. Paris 60, 70–74 (1865) = Oeuvres, vol. 2, Paris, 1905, pp. 18–22, theorem VIII.]

B. Segre (Manchester).

Motzkin, Th. The hypersurface cross ratio. Bull. Amer. Math. Soc. 51, 976–984 (1945). [MF 14472]

Given $n+1$ quantics (or forms) in $n+1$ variables $x_0, x_1, \cdots, x_n$, say $a_0, a_1, \cdots, a_n$, the author denotes by $[a_0, a_1, \cdots, a_n]$ their resultant. He also considers the set formed by the points P_i of intersection of the n primals $a_1=0, \cdots, a_n=0$ of the n-dimensional space where the x's are homogeneous coordinates, taken with proper positive multiplicities λ_i; the arbitrary factors in the coordinates of these points can be chosen in such a way that

$$(1)\qquad [a_0, a_1, \cdots, a_n]=\prod_i a_0(P_i)^{\lambda_i}$$

and then the intersection set is denoted by $[a_1, \cdots, a_n]$. The notion of resultant is then extended to the case in which the a's are quotients of two quantics, say $a_k=b_k:c_k$, by assuming

$$(2)\qquad [a_0, a_1, \cdots, a_n] = \prod[d_0', d_1', \cdots, d_n']/\prod[d_0'', d_1'', \cdots, d_n''],$$

where either $d_k'=b_k$, $d_k''=c_k$ or $d_k'=c_k$, $d_k''=b_k$ $(k=0, \cdots, n)$ and the two products involve all possible arrangements with an even or odd number of c's, respectively. In particular, if $n=1$ and b_0, c_0, b_1, c_1 are linear forms, the expression (2) reduces to the cross-ratio of the roots of these forms, so that (2) can be considered as the generalized cross-ratio of the n pairs of primals $b_k=0$, $c_k=0$. The notion of intersection $[a_1, \cdots, a_n]$ can be extended to the case of rational functions $a_k=b_k:c_k$, by allowing the multiplicities λ_i of its points to be positive or negative, in such a way that (1) continues to hold.

The author investigates the behaviour of resultants, intersections, generalized cross-ratios and intersections with respect to rational transformation of the x's, permutation of the a's or of the b's and c's, linear combination of the a's, variation of a_k, b_k, c_k into $a_k+\alpha_{ki}a_i$, $b_k+\beta_{ki}b_ic_i$, $c_k+\gamma_{ki}b_ic_i$, respectively. He also discusses the possibility for a generalized cross-ratio $[a_0, a_1, a_2]$ to be independent of a_0.

B. Segre (Manchester).

Motzkin, Th. The pentagon in the projective plane, with a comment on Napier's rule. Bull. Amer. Math. Soc. 51, 985–989 (1945). [MF 14473]

The present paper, after having recalled the well-known relations connecting the six values of the cross-ratio of four collinear points taken in different orders, and given the additional relations for the cross-ratios of any four of five collinear points, investigates the relations for the cross-ratios of any four collinear points among the intersections of five general coplanar lines. The vertices of a plane pentagon are on a conic, the sides touch another conic, and the pentagon is self-polar with respect to a third conic C, each vertex being the pole of the opposite side. The case when C has no real points is characterized; then the relations among the cross-ratios can be interpreted in the elliptic geometry defined by C, and turn out to be equivalent to the rule of Napier on the five parts of a rectangular spherical triangle. The intersections of the diagonals of the pentagon determine another pentagon, projective to the given one, and the reiteration of this process leads to an infinite sequence of pentagons, converging to the vertices and sides of the common self-polar triangle of the three conics mentioned above. Finally, the relations among the generalized cross-ratios [cf. the preceding review] of six coplanar points or lines are obtained. [The cross-ratios of five coplanar points or lines were considered first by A. F. Möbius, Der Barycentrische Calcul, Leipzig, 1827, §§190–191 = Gesammelte Werke, vol. 1, Leipzig, 1885, pp. 228–231. Interpretations in both elliptic and hyperbolic geometry have already been given by W. Fenchel, Mat. Tidsskr. B. 1941, 18–30 (1941); these Rev. 3, 13.]

B. Segre (Manchester).

Rossier, Paul. Sur une construction relative à la perspective d'un cercle. C. R. Séances Soc. Phys. Hist. Nat. Genève 61, 16–17 (1944). [MF 14221]

Generalizing, by projection, a known property of the circle [A. Mannheim, Cours de Géométrie Descriptive de l'École Polytechnique, Paris, 1880, p. 43] the author obtains the following proposition. If the lines a, b, c touch a

conic (S) in the points A, B, C, and a transversal through $D=ab$ meets AB, AC, BC and c in K, L, M and N, the point $P=(AM, BL)$ lies on (S), and the tangent to (S) at P passes through N. A direct proof of the proposition is also given. *N. A. Court* (Norman, Okla.).

Rossier, Paul. Application à la théorie de l'inversion d'un système de coordonnées dû à Varignon. C. R. Séances Soc. Phys. Hist. Nat. Genève 59, 134–138 (1942). [MF 14205]

Rossier, Paul. Sur une réciproque d'un théorème de Darboux relatif aux courbes anallagmatiques. C. R. Séances Soc. Phys. Hist. Nat. Genève 60, 108–110 (1943). [MF 14216]

The author proves that if a curve of degree $2p$ is anallagmatic (that is, invariant under inversion) its study may be reduced to the study of a curve of degree p. Use is made of a system of polar coordinates due to Pierre Varignon (1654–1722), considered by the author in the note quoted above. *N. A. Court* (Norman, Okla.).

de Cesare, E. A. Synthetic theory of imaginary elements according to the method of C. Segre. An. Soc. Ci. Argentina 140, 177–221 (1945). (Spanish) [MF 14095]

This paper gives a detailed account of the well-known manner of introducing and using pairs of complex conjugate elements in synthetic projective geometry, by considering them as the double elements of an elliptic involution. It follows in the main C. Segre [Mem. Accad. Sci. Torino (2) 38, 3–24 (1886)], dealing essentially with a number of properties of projectivities upon a line and with a few results on conics. *B. Segre* (Manchester).

Fabricius-Bjerre, Fr. On the normals of quadric surfaces in a non-Euclidean space. Mat. Tidsskr. B. 1945, 75–80 (1945). (Danish) [MF 14263]

Let ϕ and α, in complex projective 3-space, be two quadrics that do not touch. With a common polar tetrahedron as coordinate tetrahedron the equations of ϕ and α may be normed so that they have the form $\phi: \sum x_i^2=0$, $\alpha: \sum a_i x_i^2=0$. Put $d_{ik}=a_k-a_i$ and assume $d_{ik}\neq 0$. The quadric ϕ is considered as the absolute locus of a non-Euclidean metric. The normals to α form a congruence N of order 6 and class 2. The lines in N which belong to a given linear complex form a regulus of order 8. The axial complex A to α consists of the lines which are perpendicular to their polar with respect to α. In line coordinates z_{ik} the equation of A is $z_{14}z_{23}d_{14}d_{23}=z_{24}z_{31}d_{24}d_{31}=z_{34}z_{12}d_{34}d_{12}$. Therefore A is a tetrahedral complex and contains the normals to all surfaces confocal with α. *H. Busemann* (Northampton, Mass.).

Nyström, E. J. Zwei Modelle zur Erläuterung des Dualitätsprinzips in der Ebene. Soc. Sci. Fenn. Comment. Phys.-Math. 11, no. 1, 8 pp. (1 plate) (1942). [MF 14630]

If A_1B_1 and A_2B_2 are segments of constant lengths on fixed lines s_1 and s_2, while A_1A_2 passes through a fixed point E, then B_1 and B_2 are projectively related to A_1 and A_2, which are in perspective from E; therefore B_1 is projectively related to B_2, and the envelope of B_1B_2 is a conic touching s_1 and s_2. Dually, if PS_1Q and PS_2Q are angles of constant sizes at fixed points S_1 and S_2, while P runs along a fixed line e, then the locus of Q is a conic through S_1 and S_2. Different positions for E yield different conics touching s_1, s_2 and a third fixed line s_3 (which is the position of B_1B_2 when A_1 and A_2 coincide). Dually, different positions for e yield different conics through three fixed points S_1, S_2, S_3. The transformation from a line e to a conic through S_1, S_2, S_3 is in fact a quadratic transformation based on the triangle $S_1S_2S_3$. The author describes a practical instrument for carrying out each of these constructions. *H. S. M. Coxeter* (Toronto, Ont.).

Nyström, E. J. Zur praktischen Axonometrie. Soc. Sci. Fenn. Comment. Phys.-Math. 11, no. 11, 25 pp. (1943). [MF 14631]

Two orthographic views of an object are placed in a plane and two families of parallel projection lines are drawn, one family through the points of each view. The points of intersection of corresponding projection lines then form an axonometric view of the object. Whether this view is oblique or orthographic depends on the positions of the projections and the projection lines. This method of constructing axonometric views was proposed by L. Eckhart [Akad. Wiss. Wien. S.-B. IIa. 146, 51–56 (1937)]. The author elaborates this procedure and formulates a condition to ascertain that the view produced is an orthographic axonometric projection. *E. Lukacs* (Cincinnati, Ohio).

Wunderlich, Walter. Eindeutigkeitsfrage der Hauptaufgabe der Photogrammetrie beim Finsterwalderschen Folgebildanschluss. Monatsh. Math. Phys. 51, 57–58 (1943). [MF 14382]

The author remarks that conditions imposed by S. Finsterwalder on the problem of photogrammetric reconstruction [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1941, 91–110 (1941)] do not exclude the possibility of multiple solutions discussed by the author [same Monatsh. 50, 151–164 (1941); these Rev. 6, 15] and by J. L. Krames [same Monatsh. 49, 327–354 (1941); 50, 1–13, 65–83, 84–100 (1941); these Rev. 6, 14, 15]. *E. Lukacs* (Cincinnati, Ohio).

Convex Domains, Integral Geometry

Jackson, S. B. The four-vertex theorem for surfaces of constant curvature. Amer. J. Math. 67, 563–582 (1945). [MF 13940]

Dans un article précédent [Amer. J. Math. 62, 795–812 (1940); ces Rev. 2, 158] l'auteur avait montré que sur toute courbe simple fermée de classe (C''') située sur une sphère sans être un cercle, la courbure géodésique présente au moins quatre extréma dits sommets géodésiques. Dans le présent article ce résultat est étendu à toute courbe fermée C de classe (C'') située sur un domaine S simplement connexe (hypothèse (a)) d'une surface Σ de courbure totale constante K (hypothèse (b)) pourvu que C ne soit pas un cercle géodésique. Le cas général de K quelconque est ramené au cas de $K=0$ (cas du plan) grâce à la proposition suivante: Il existe une transformation T appliquant S sur un domaine plan, laquelle est localement biunivoque, conserve l'orientation et fait correspondre des cercles géodésiques du plan (de Gauss) aux cercles géodésiques de S. Des contre-exemples sont donnés montrant que chacune des hypothèses (a) et (b) est indispensable [Le référent pense qu'on peut se dispenser de l'hypothèse (a) si l'on exige que la courbure géodésique de C ne s'annule pas.] *C. Pauc.*

Blanc, E. Les ensembles surconvexes plans. Ann. Sci. École Norm. Sup. (3) 60, 215–246 (1943). [MF 14642]

Let A and B be two points of the plane, and $2R>AB$. The closed "lens" of radius R defined by A and B, written

$L(A, B; R)$ is the set of points common to the two closed circles of radius R through A and B. A closed set E is said to be superconvex for R, or scv$R$, if for every A, B contained in E such that $AB < 2R$, $L(A, B; R)$ is also contained in E. In the present paper the author develops systematically and in detail the properties of scv$R$ sets. It follows immediately from the definition that an scv$R$ set either is connected or its various components are distant by more than $2R$. Since the definition imposes no relationship between two such components, all attention is paid to connected sets. A connected scv$R$ set is convex and so enjoys all the properties of convex sets, and in addition numerous properties bearing a relation to circles of radius R analogous to that which corresponding properties of convex sets bear to lines. Among the properties of connected scv$R$ sets E are the following: (1) E is contained in a circle of radius R; (2) the lower curvature in the sense of Menger of the boundary of E is not less than $1/R$; (3) through every point of the boundary of E there pass one or more R-circles of support, that is, circles of radius R containing E; (4) E is the set common to all R-circles containing E.

Two notions used extensively in the investigations are those of the R-front and the R-adjoint RE of an arbitrary set E. The R-front of E is the set of boundary points of E through which pass R-circles of support of E; the R-adjoint of E is the set common to all R-circles with centers in E. A study is made of the relation between the R-front of E and the boundary of RE. It is shown that, if E is scv$R$, then RRE is equal to E; otherwise RRE is defined to be the R-hull of E and shown to be the intersection of all R-circles containing E, if such exist.

The final portion of the paper is devoted to proving the following theorem. If E is a continuum which is of order 2 with respect to the family $\mathfrak{F}_R$ consisting of all lines and circles of radius more than R, then E is an arc of the boundary of a set which is scv$R$. This is a generalization of a theorem of A. Marchaud [Acta Math. **55**, 67–115 (1930)] on continua of order 2. *J. W. Green.*

Blanc, Eugène. Correspondance entre les frontières d'un ensemble et de son R-adjoint. Ensembles R-complémentaires. C. R. Acad. Sci. Paris **220**, 74–76 (1945). [MF 15138]

Let E be a set in the plane, F its boundary, RE its R-adjoint, F' the boundary of RE, K the R-hull of E, $\mathfrak{F}$ the boundary of K, and Φ the R-front of E. The author gives a simpler proof of theorem XVII of the paper reviewed above, namely, that $\mathfrak{F}$ is a closed Jordan curve and that the portion of $\mathfrak{F}$ complementary to Φ consists of a finite or denumerable set of arcs of R-circles. In addition, he shows that if, at a point of F, the lower curvature of F in the sense of Menger is $1/r$, then the upper curvature of F' at the corresponding point is $1/(R-r)$. Finally, he takes up the subject of the effect on F' of deforming F infinitesimally. It is remarked that such a deformation in the neighborhood of a point of F can result in the deformation of a finite arc of F', although the new position of F' differs infinitesimally from the old. *J. W. Green* (Los Angeles, Calif.).

Dinghas, Alexander. Über die lineare isoperimetrische Ungleichung für konvexe Polygone und Kurven mit Ecken. Monatsh. Math. Phys. **51**, 35–45 (1943). [MF 14380]

Let K be a convex body in two dimensions. If F is the area of K, L its circumference, and d the semibreadth in any direction, then (1) $F-Ld+\pi d^2\leqq 0$ or $F\leqq L^2/4\pi-\pi(L/2\pi-d)^2$ is a well-known sharpening of the isoperimetric inequality. In the present paper the author obtains a similar inequality (2) $F-Lh+F^*h^2\leqq 0$, where h is the maximum distance of points of K from the boundary of K and F^* is the area of the circular cap-body K^* of K. This latter is defined as follows. If T is a directed tangent to K, a tangent T' to the unit circle C is constructed having the same direction as T and the same sense of rotation about C as T had about K. Then if $K(T')$ is the half-plane determined by T' and containing C, K^* is the set common to all $K(T')$. Unless K has corners, $F^*=\pi$. If K has corners with angles ψ_i, then $F^*=\pi+\sum(\tan\frac{1}{2}\psi_i-\frac{1}{2}\psi_i)$. The equality sign in (2) is shown to hold only if K is a circular cap-body or if K is a cap-body constructed in a similar fashion on a figure consisting of a rectangle with opposing semicircular caps. *J. W. Green* (Los Angeles, Calif.).

Dinghas, Alexander. Verschärfung der Minkowskischen Ungleichungen für konvexe Körper. Monatsh. Math. Phys. **51**, 46–56, 56a (1943). [MF 14381]

For a convex body K in three dimensions, let σ be the surface area, V the volume, and M 4π times the average distance of the supporting planes from an interior point. From the Brunn-Minkowski inequalities for mixed volumes come the two inequalities: (1) $\sigma^2\geqq 3MV$, (2) $M^2\geqq 4\pi\sigma$, which combine to give the isoperimetric inequality. The equality in (2) holds only for spheres, while that in (1) holds for spheres and also for spherical cap-bodies. A spherical cap-body is a sphere to which has been added nonintersecting tangent circular conical caps. The author seeks to replace (2) by a similar inequality in which the equality sign will occur for spherical cap-bodies as well as for spheres. This is done by defining for any convex body K an associated cap-body K^* as follows: the exterior normal unit vectors to K when transferred to the origin map on a set S of the unit sphere Σ. The interior points of $\Sigma-S$ fall into a finite or denumerable set of distinct regions R_i on Σ; in each R_i the maximum circle is constructed and the tangent circular conical cap constructed along the circle; the resulting body is K^*. Unless K has corners, K^* is the unit sphere. If σ^* is the area of K^*, then the new inequality obtained in place of (2) is $M^2\geqq\sigma^*\sigma$, the equality holding for spherical cap-bodies. *J. W. Green* (Los Angeles, Calif.).

Hadwiger, H. Die erweiterten Steinerschen Formeln für ebene und sphärische Bereiche. Comment. Math. Helv. **18**, 59–72 (1945).

Let A_ρ denote an outer parallel region of a closed plane set A, that is, the set of points within distance ρ of A, and $A_{-\rho}$ the corresponding inner parallel region. The author extends Steiner's formulae for the area $F(\rho)$ and perimeter $L(\rho)$ of A_ρ to a wider class than that of convex regions. He supposes (i) that A is "subconvex of degree α," that is, α is the least number such that $A\cdot K_\rho$ is simply connected for every disc K_ρ of radius $\rho<\alpha$; (ii) that A is "superconvex of degree β," that is, the closure of its complement is subconvex of degree β. (A convex region is subconvex of degree ∞.) Furthermore, (iii) let A consist of n separate pieces and have m rectifiable bounding continua. Then $F(\rho)=F+L\rho+(2n-m)\pi\rho^2$, $L(\rho)=L+(4n-2m)\pi\rho$, for $-\beta\leqq\rho\leqq\alpha$, where $F=F(0)$, $L=L(0)$. Elimination of ρ yields the parallel-invariance of $L^2-(8n-4m)\pi F$, which generalizes the isoperimetric deficit of a convex region. The author finds it convenient to obtain these formulae as the limiting case of those for sets on a sphere.

H. P. Mulholland (Beirut).

Hadwiger, H. Über Integralmittelwerte bei geschlossenen sternförmigen Kurven. Vierteljschr. Naturforsch. Ges. Zürich 87, 199–203 (1942). [MF 13967]

Let $R=R(\varphi)$ be the polar equation of a closed plane curve C and $\theta(\varphi)$ the (positive) acute angle between tangent and radius vector, $R(\varphi)$ and $\theta(\varphi)$ being everywhere one-valued, continuous, and positive. Let $[\mathfrak{M}_\alpha(f)]^\alpha$ denote the mean value of $[f(\varphi)]^\alpha$ over $0\leqq\varphi\leqq 2\pi$. The author obtains, from well-known properties of $\mathfrak{M}_\alpha$, various inequalities involving $R_\alpha=\mathfrak{M}_\alpha(R)$, $S_\alpha=\mathfrak{M}_\alpha(\operatorname{cosec}\theta)$, the length L of C, and the area F that it encloses. The following are specimens:

$$R_\alpha\leqq\sqrt{(F/\pi)},\qquad -\infty<\alpha\leqq 2;$$
$$2\pi R_\lambda S_\mu\geqq L,\quad \pi R_\lambda R_\mu\geqq F,\qquad \lambda>1,\ \lambda^{-1}+\mu^{-1}=1;$$
$$L^2-4\pi F\leqq 4\pi^2R_2^2(S_2^2-1)=4\pi F(S_2^2-1).$$

H. P. Mulholland (Beirut).

Differential Geometry

Roeser, Ernst. Konforme Abbildung der hyperbolischen Ebene und der Kugel auf die Grenzkugel. Deutsche Math. 5, 299–305 (1940). [MF 14335]

The author's preface reads as follows. Es soll gezeigt werden, wie man die nichteuklidische Geometrie positiver und negativer Krümmung auf eine Fläche mit euklidischer Metrik abbildet, ohne sich projektiver Massbestimmung zu bedienen. Die Darstellung soll so gewählt werden, dass der Unterschied zur Kleinschen Abbildung deutlich hervortritt, indem diese schrittweise in die neue Abbildung übergeführt wird. *M. H. Heins* (Cambridge, Mass.).

Nyström, E. J. Abwicklung von Kegelflächen zweiter Ordnung. Soc. Sci. Fenn. Comment. Phys.-Math. 12, no. 12, 11 pp. (1944). [MF 14629]

If a quadric cone is developed onto the plane, the central angle ω of the resulting circular sector can be given by an elliptic integral. The author obtains an approximation formula and a table for ω, and also describes methods for determining the focal axes and planes of symmetry of the cone. *S. B. Myers* (Ann Arbor, Mich.).

Combes, Bernard. Une formule de géométrie sphérique et son application au calcul de l'aire d'une surface gauche de paramètre de distribution constant. C. R. Acad. Sci. Paris 218, 926–927 (1944). [MF 15241]

Let E and E_1 be two closed curves on the unit sphere which intercept arcs of constant length δ on the great circles making a fixed angle V with E. By using the Gauss-Bonnet formula the author obtains the relation

$$2\pi-a_1=L\sin V\sin\delta+(2\pi-a)\cos\delta,$$

where L is the length of E and a and a_1 are areas bounded by E and E_1. This formula is then applied to show that the spherical image of a closed line of striction on a non-developable ruled surface divides the sphere into two equal parts and to establish other allied results. The work is closely related to a theorem of Jacobi [Gesammelte Werke, vol. 7, Berlin, 1891, p. 39] which is mentioned. [Both the author and Jacobi appear to ignore the fact that in the applications the curves E and E_1 are frequently not simple and that in this case the terms a and a_1 of the formula require more careful definition.] *S. B. Jackson.*

Richmond, Herbert W. On minimal surfaces. J. London Math. Soc. 19, 229–241 (1944). [MF 14573]

The author outlines the advantages of studying minimal surfaces by use of tangential plane coordinates. Let (L, M, N, P) denote numbers proportional to the direction cosines of the normal and the normal distance of a plane in ordinary space. A surface may be defined by a homogeneous equation of first degree, $P=F(L, M, N)$. The parametric equations in Cartesian coordinates are obtained from this by differentiating P with respect to L, M, N, respectively. Thus each coordinate of a point on the surface is a homogeneous function of degree zero in (L, M, N). It is proved that the surface is minimal if and only if the homogeneous function P of first degree satisfies the Laplace equation. This result was noted by Bromwich [Proc. London Math. Soc. 30, 276–281 (1899)]. From this, an orderly systematic classification of the various well-known types of minimal surfaces can be achieved.

The author's general solution is compared with Weierstrass's solution of the Lagrange equation defining minimal surfaces and also with the methods of Monge and Sophus Lie, both of whom employed parametric representations of minimal curves. The catenoid, helicoid and the minimal surfaces of Enneper and Henneberg are obtained in tangential plane coordinates. Classes of minimal surfaces are given which include the preceding types as special cases. Finally, there is a full discussion of the problem of finding the minimal surface which is divided symmetrically by a plane and cuts the plane orthogonally in a known oval curve.

J. DeCicco (Chicago, Ill.).

Ghosh, N. N. Matrix treatment of a rigid body motion in complex space. Bull. Calcutta Math. Soc. 37, 43–50 (1945). [MF 13781]

A vector in n-space is here represented by a square matrix of order $n+1$ in which the n last places of the first row are occupied by what could be called the components of the vector and the last n places of the first column are the complex conjugates of these quantities, the remaining elements of the matrix being zero. Generalizing the discussion in an earlier paper [same Bull. 32, 109–120 (1940); these Rev. 3, 191], where the components were real, the author defines operations on these vectors in terms of matrix operations and introduces other special types of Hermitian matrices. Using these he builds a theory of a "rigid body," discussing its velocity, accelerations, center of mass, the inertia matrix, kinetic energy, etc., all in close analogy with the theory for the real case of his earlier paper (all scalar quantities here also have real values). One point of difference with that theory is that here to every vector there corresponds an adjoint vector and the plane determined by the two plays a certain part in the theory of rotations; and, of course, the difference between symmetric and skew-symmetric matrices becomes trivial. *G. Y. Rainich.*

Haag, J. Sur quelques mouvements remarquables. Bull. Sci. Math. (2) 69, 39–45 (1945). [MF 14426]

Let P be a point fixed relatively to an orthogonal frame of reference R. Let H_1, H_2 be two helicoidal movements of R. To each point M there corresponds by each H_i a velocity vector V_i. The locus of M such that P lies in the plane of V_1, V_2 through M is a surface Ω, which is a cyclide of the third order. The same surface is obtained if H_1, H_2 are replaced by $\bar{H}_1$, $\bar{H}_2$ of a group g_2 defined by $\bar{H}_\alpha=\lambda_\alpha{}^iH_i$ $(\alpha, i=1, 2)$. Whatever H_2 may be, M lies on Ω if the vector

V_1 through M passes through P. The locus of M is a skew cubic curve γ. There is then an infinity of such cubics γ on Ω. Let there be a third movement H_3 not belonging to g_2, determining with H_1, H_2 a group g_3. Let $\bar{H}_\alpha=\lambda_\alpha{}^i H_i$ ($\alpha=1, 2; i=1, 2, 3$) be any two movements of g_3. The surfaces $\bar{\Omega}$ defined by $\bar{H}_\alpha$ form a net of surfaces all of which have a unicursal quartic curve in common. *V. G. Grove.*

Springer, C. E. Rectilinear congruences whose developables intersect a surface in its lines of curvature. Bull. Amer. Math. Soc. 51, 990–996 (1945). [MF 14474]

Let Λ be a rectilinear congruence in ordinary three dimensional space. The intersections of the developables of Λ with an arbitrary surface S constitute a net of curves which is termed the "intersector net." The author determines the condition under which the intersector net coincides with the lines of curvature or asymptotic lines of S. *A. Fialkow.*

Charrueau, André. Sur la déformation infiniment petite et sur des congruences qui s'y rattachent. C. R. Acad. Sci. Paris 219, 107–108 (1944). [MF 14510]

Let the surfaces S and S_1 through points (x, y, z) and (x_1, y_1, z_1), respectively, correspond with orthogonality of linear elements. Let S' and S_1', through points (x, y_1, z) and (x, y, z_1), respectively, be in correspondence in a similar manner. Four rectilinear congruences are defined relative to these surfaces. Various geometric properties which involve these congruences and the twelve surfaces of Darboux corresponding to S, S_1 and to S', S_1' are stated. *A. Fialkow.*

Charrueau, André. Sur des congruences de droites déduites d'une même surface. C. R. Acad. Sci. Paris 221, 274–276 (1945). [MF 14496]

Let S be a nondevelopable surface, A any point of S, Oz any ray and H the plane through O orthogonal to Oz. The projection of A upon H is rotated through $+\pi/2$ about Oz to a point through which a line D is drawn parallel to the normal to S at A. As A is displaced upon S, D generates a congruence C. Another congruence E is defined in a somewhat analogous manner. Various geometric properties of C and E are announced of which the following is typical: the developables of C (or of E) correspond to the asymptotic lines of S. *A. Fialkow* (New York, N. Y.).

Vincensini, Paul. Sur une famille de congruences à angle des plans focaux constant. Ann. Fac. Sci. Univ. Toulouse (4) 6, 1–6 (1943). [MF 15170]

The author considers line congruences for which the angle between the focal planes is constant. He gives a purely geometric description of all these congruences. Going to a degenerate case, he considers pairs of curves with the property that the angle under which they appear to cross each other, when viewed from an arbitrary point, is constant. He shows that this can happen only if the angle is 90° and the curves are two focal conic sections. *H. Samelson.*

Vincensini, Paul. Sur une relation entre les congruences isotropes et les congruences d'Appell. Bull. Sci. Math. (2) 69, 52–62 (1945). [MF 14428]

A correspondence between isotropic congruences and congruences of Appell was established earlier by the author [same Bull. (2) 65, 155–178, 186–210 (1941); these Rev. 7, 31]. In the present paper he develops the consequences of this relationship with regard to (1) congruences of lines paratactic to the same circle; (2) congruences of Appell and their focal nappes; (3) certain congruences of circles defined by a property of the system of their focal points. *C. B. Allendoerfer* (Haverford, Pa.).

Vincensini, Paul. Sur les congruences des cordes de contact des enveloppes de sphères. Ann. Sci. École Norm. Sup. (3) 61, 119–147 (1944). [MF 14647]

The congruences of the chords joining the points of contact of the envelopes of two-parameter systems of spheres were introduced by Ribaucour. The author presents a detailed study of them. An arbitrary rectilinear congruence can always be considered as a congruence of chords of contact of envelopes of spheres. Thus the equations of any rectilinear congruence can be placed in an invariant form analogous to that obtained by Weingarten for the equations of a normal rectilinear congruence. Applications are made to related transformations and to a net of curves discovered by Drach. The normal congruences of chords of contact relative to any arbitrary central locus depend on a partial differential equation of second order, related to that of Laplace. This equation is satisfied if the central locus is a sphere. By deformation of the spherical central locus, normal congruences of chords of contact are obtained for which the central locus is of constant Gaussian curvature.

The author studies in detail the general problem of finding the envelopes of spheres defined by normal congruences of chords of contact, which remain normal and of contact by a deformation, finite, continuous or arbitrary, of the central locus. There exist congruences of spheres deformable with the conservation of orthogonality of the chords of contact which belong to each of these three types of deformations. The case of finite deformation leads to the choice of pairs of central loci belonging to the class of surfaces having the same spherical representation of their lines of curvature as the deformations of the sphere. The case of arbitrary deformation leads to surfaces applicable upon a surface of revolution. The problem of finding the envelopes of spheres of normal congruences of chords of contact is intimately related to many questions concerning triply orthogonal systems and cyclic systems. The examination of these questions from the point of view adopted in this paper leads, in particular, to transformations of normal cyclic systems. *J. DeCicco* (Chicago, Ill.).

Vincensini, Paul. Sur les courbes de Ribaucour et sur les réseaux conjugués géodésio-coniques. Ann. Sci. École Norm. Sup. (3) 60, 17–34 (1943). [MF 14637]

Let a surface (A) be the envelope of a one parameter family of spheres with the following property: if M and N are any two corresponding centers of principal curvature of (A), the locus of points I dividing the segment MN in a constant ratio K is a plane. In a previous paper [same Ann. (3) 59, 141–164 (1942); these Rev. 6, 187] the author has studied the case where I is the midpoint of MN ($K=-1$). In the present paper the case of a general value for K is investigated. If the spheres all have the same radius, the surface (A) is a canal surface and the curve C which is the locus of their centers is a curve of Ribaucour. If the radius of the spheres is allowed to vary, the curve C is a plane curve whose equation depends upon a parameter a. If $a>1$, the curve C is the projection of a curve of Ribaucour onto a plane which makes an angle $\theta=\sin^{-1}(1/a)$ with the plane of the curve of Ribaucour. If $a<1$, a similar transformation involving imaginaries leads from a curve of Ribaucour to the curve C. When $a=1$, C has no connection with a curve of Ribaucour, but has other interesting properties. Supposing

that M lies on the curve C, the point N describes a surface Σ, the "second" nappe of the involute of (A). The surface Σ possesses a special type of conjugate net called a "geodesic-conic conjugate net," whose properties are investigated.

C. B. Allendoerfer (Haverford, Pa.).

Sprague, Atherton H. Surfaces whose lines of curvature are nets R, and their transformations. II. Revista Ci., Lima 47, 135–172 (1945). [MF 14482]

Continuing part I [same Revista 47, 3–32 (1945); these Rev. 7, 32] the author studies F transforms of the lines of curvature of a surface S which is an R net into nets of the same kind on a surface S'. Analytical conditions on S and S' and the parameters of the transformation F are found. Among such surfaces are those with constant negative Gaussian curvature.

V. G. Grove (East Lansing, Mich.).

MacQueen, M. L. Conjugal quadrics and the quadric of Moutard. Amer. J. Math. 68, 161–172 (1946). [MF 15498]

A general member of the two parameter family of conjugal quadrics of Grove [Bull. Amer. Math. Soc. 50, 231–234 (1944); these Rev. 5, 216] associated with a conjugate net N_λ at a point P_z of a surface S and the quadric of Moutard for the direction λ at the point intersect in the asymptotic tangents through P_z and in a conic which lies in a plane π_λ through P_z. The author obtains equations in local plane coordinates for the envelope of π_λ as λ varies. This envelope, called the cone (2.10), is found to be a cone of the sixth class with its vertex at P_z, corresponding to two parameters, h, L. The third polar of the tangent plane of S at the point P_z with respect to the cone (2.10) is found to be a cone (3.1) of class three whose equations are independent of L. The cusp-axis of this cone, if h is not equal to one of the values 1, $\frac{1}{3}$, $-\frac{1}{3}$, is a canonical line $l_1(k)$ which yields, by proper choice of h, any line of the canonical pencil. The second polar of the tangent plane of S at P_z with respect to the cone (2.10) is a cone (3.3) of the fourth class which varies with L. Through each asymptotic tangent at the point P_z of S there is a plane which is tangent to all of the cones (3.3) with variable L. These two planes intersect in a general canonical line $l_1(k)$.

The author obtains the results which he summarizes as follows. "There are two planes p_1, p_2 besides the tangent plane of the surface S, through the tangent line in the direction λ at the point P_z, which are tangent to the cone (3.3). The plane p^* which is the harmonic conjugate of the tangent plane $z=0$ with respect to the planes p_1 and p_2 envelopes the cone (3.1) in case any one of the following conditions holds: (i) the conjugal quadric (2.3) is a quadric of Darboux; (ii) the direction of the curve C is a direction D^*_k; (iii) the curve C_λ is a curve of Segre which is tangent to the first canonical tangent at P_z." The verification of the following theorem is left to the reader: the three planes p^*_i $(i=1, 2, 3)$, each of which is a plane p^* which corresponds to a direction of Segre at P_z, are concurrent in the canonical line $l_1(k)$.

The remainder of the paper is devoted to a brief study of special cases, which yields geometric characterizations of unique conjugal quadrics for which $h=0$, $\frac{2}{3}$, $\frac{1}{3}$, $-\frac{1}{3}$, 1 and, associated with them, two new interpretations of the directrix of Wilczynski, a new interpretation of the edge of Green, and a new characterization for $l_1(k)$ corresponding to each of the following values: $k=-\frac{7}{12}$, $-\frac{3}{4}$, $-\frac{1}{6}$, $-\frac{1}{12}$.

P. O. Bell (Lawrence, Kan.).

Su, Buchin. On the projective differential geometry of a non-holonome surface in ordinary space. Ann. Mat. Pura Appl. (4) 19, 289–313 (1940). [MF 15089]

This paper generalizes for the case of a nonholonomic surface the following configurations and correspondences of projective surface theory and discusses certain geometrical interpretations: the polarities of Lie and of Palozzi, cone of Calapso, principal quadrics of Lane and Bompiani, edges of Green, axis of Čech, quadrics of Darboux and of Moutard, correspondences of Moutard and of Segre. Some of these are offered as alternatives to previous definitions. The correspondences which generalize the polarities of Lie and Palozzi are no longer polarities themselves but lie in a one parameter family of correspondences one of which is still a polarity in the nonholonomic case. The pencil of quadrics associated with this polarity furnishes the generalization of the principal quadrics of Lane. [See the following review.]

J. L. Vanderslice (College Park, Md.).

Bortolotti, Enea. Sulla geometria proiettiva differenziale di una superficie anolonoma nello spazio ordinario. Ann. Mat. Pura Appl. (4) 19, 315–325 (1940). [MF 15090]

The author, having revised the proofs of the paper of Su reviewed above, further complements it with the present note. He clarifies certain portions of it, particularly those on the axis of Čech and the quadrics of Moutard, with discussion and some extension of past work by himself and others. Several new problems are suggested.

J. L. Vanderslice (College Park, Md.).

Matsuda, Mitsuo. On the real representation of imaginary elements. Jap. J. Math. 18, 323–377 (1942). [MF 14971]

The author considers a Lie group G of point transformations involving $(rn+n-r)$ real essential parameters in space of n dimensions. It is assumed that G has the property that there is one and only one transformation T which converts a complex differential element e of order r into an element e_0 whose real parts are all zero. This pure imaginary element e_0 can be represented by a pair of real elements whose corresponding coordinates are of opposite signs and whose numerical values are the absolute values of the corresponding ones of e_0. The inverse T^{-1} of T converts these into two real elements E and E'. These are said to constitute a real representation of the complex differential element e. If T is any transformation of G carrying a complex element e into another one e^*, then $T(E)=E^*$ and $T(E')=E'^*$. In this sense, the pair of real elements (E, E') is said to be invariant under the group G.

Applications of the theory are made to the lineal elements under the motion group of the Euclidean and hyperbolic planes. Real representations of analytic curves are obtained. The projective group on a complex line is also considered from this point of view. The real representations of analytic curves in Euclidean space lead to two curves on a pair of surfaces which have many interesting interrelationships. For example, any minimal curve is pictured by a pair of real curves which are contained in a pair of applicable minimal surfaces, the tangent lines at corresponding points being perpendicular. The study of imaginary points in Euclidean space leads to a consideration of circles orthogonal to a sphere. Finally, the author considers from this point of view the various relationships between a pair of quadrics in space.

J. DeCicco (Chicago, Ill.).

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen konformen, hyperbolischen konformen und parabolischen konformen Differentialgeometrien. II. Proc. Imp. Acad. Tokyo 17, 330–338 (1941). [MF 14716]

[Part I appeared in the same Proc. 16, 333–340 (1940); these Rev. 2, 166.] The author considers a binary complex variable $z=x+jy$, where $j^2=\mu+j\nu$. The binary variable z is said to be hyperbolic or elliptic according as $(\nu^2+4\mu)$ is positive or negative. It is parabolic if $(\nu^2+4\mu)^{\frac{1}{2}}$ is an infinitesimal. By the introduction of a modulus $|z|$ and an amplitude θ, analogous to the ordinary complex variable case, a trigonometry is derived. An angle between two curves is defined. The author next considers the j-Moebius group, represented by linear fractional functions of the binary variable z. These leave unchanged a certain family of conics (hyperbolas or ellipses or parabolas). By generalizing the concepts of tetracyclic and pentaspherical coordinates, the author obtains the fundamental invariants of the j-Moebius group, not only in the plane but also in space. This group is conformal in the sense that the angle between two curves, defined by this binary variable, is left unchanged. [In connection with the binary complex variable z, P. F. Capelli studied not only holomorphic but also polygenic functions. See An. Soc. Ci. Argentina 128, 154–174 (1939); Bull. Amer. Math. Soc. 47, 585–595 (1941); these Rev. 1, 300; 3, 85. Capelli found a variation of the derivative circle theorem of Kasner, Science (N.S.) 66, 581–582 (1927).]

J. DeCicco (Chicago, Ill.).

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen Laguerreschen, hyperbolischen Laguerreschen und parabolischen Laguerreschen Differentialgeometrien. II. Proc. Imp. Acad. Tokyo 17, 339–343 (1941). [MF 14717]

[Part I appeared in the same Proc. 16, 346–349 (1940); these Rev. 2, 166.] The author obtains a line geometry, roughly the dual of that outlined in the preceding review. Using the notions of distance and angle developed previously, a set of homogeneous line coordinates (also plane coordinates in space) are defined. The author studies a group of line (plane) transformations which leave invariant a family of conics (quadrics). This is a variation of the Laguerre group of circular line (spherical plane) transformations. The fundamental invariants of these new j-Laguerre groups are found. *J. DeCicco* (Chicago, Ill.).

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen Lieschen, hyperbolischen Lieschen und parabolischen Lieschen Differentialgeometrien. II. Proc. Imp. Acad. Tokyo 17, 344–348 (1941). [MF 14718]

[Part I appeared in the same Proc. 16, 341–345 (1940); these Rev. 2, 166.] The author continues the work outlined in the two preceding reviews. He discusses the j-Lie group of all lineal-element transformations preserving a certain family of conics. Some of its fundamental invariants are obtained. This is an analogue of the Lie circular group in the plane. These concepts are extended to space.

J. DeCicco (Chicago, Ill.).

Yano, Kentaro, et Takano, Kazuo. Sur les coniques dans les espaces à connexion affine ou projective. I. Proc. Imp. Acad. Tokyo 20, 410–417 (1944). [MF 14906]

In an affinely connected space with coordinates x^i, $i=1, \cdots, n$, a curve is called a plane curve if its development on the tangent affine space is a plane curve. With respect to a special parameter s of affine character the differential equation of the plane curves is shown to be $\delta^3x^i/ds^3=k\,dx^i/ds$, in which δ/ds designates the covariant derivative along the curve with respect to the affine connection Γ^i_{jk} and k is a function of s. The direction δ^2x^i/ds^2, therefore, serves to define an affine normal of a plane curve in an affinely connected space.

The authors define a conic in an affinely connected space to be a curve whose affine normals are concurrent. A necessary and sufficient condition for this is that k is a constant. The affine conics are classified as ellipses, parabolas or hyperbolas according as $k>0$, $k=0$ or $k<0$. A space of projective connection is defined in association with the "repère semi-naturel" of É. Cartan and a conic in this space is defined to be a curve described by a point A_0 whose homogeneous coordinates are quadratic functions of a projective parameter t. Differential equations are obtained which furnish a concept of the nature of the projective parameter t of the conics. [A subscript k has been omitted from the symbol in line 4, p. 410.] *P. O. Bell.*

Yano, Kentaro, et Takano, Kazuo. Sur les coniques dans les espaces à connexion affine ou projective. II. Proc. Imp. Acad. Tokyo 20, 418–424 (1944). [MF 14907]

[Cf. the preceding review.] In this note the authors investigate the transformations of the hyperplane at infinity which cause the projective conics to coincide with the affine conics. Let $(A_0, A_1, \cdots, A_n)$ denote the "mobile repère" attached to the point A_0, the hyperplane at infinity being determined by the points A_i, $i=1, 2, \cdots, n$. Let the components of the projective connection be denoted by Π^0_{jk} and Π^i_{jk}. The projective transformation $\overline{\Pi}^i_{jk}=\Pi^i_{jk}+\delta^i_j\varphi_k+\delta^i_k\varphi_j$, $\overline{\Pi}^0_{jk}=\Pi^0_{jk}+\varphi_{j;k}-\varphi_j\varphi_k$, in which the semicolon designates covariant differentiation with respect to the components Π^i_{jk}, accompanies the change of hyperplane at infinity defined by $\bar{A}_0=A_0$, $\bar{A}_j=\varphi_jA_0+A_j$. The following theorems are proved. (1) In order that the projective conics become the affine conics, it is necessary and sufficient that there exist a covariant vector φ_j such that $\overline{\Pi}^0_{jk}=0$. The projective conics are, then, the affine conics with respect to the affine components $\overline{\Pi}^i_{jk}$. (2) The projective transformation defined by $\overline{\Pi}^i_{jk}=\Pi^i_{jk}+\delta^i_j\varphi_k+\delta^i_k\varphi_j$, with φ_j satisfying $\varphi_{j;k}-\varphi_i\Pi^i_{jk}=\varphi_j\varphi_k$ changes the affine conic with respect to Π^i_{jk} to an affine conic with respect to $\overline{\Pi}^i_{jk}$. [The authors might have observed that by viewing the affine conic as a projective conic with respect to the projective connection Π^0_{jk} and Π^i_{jk} (in which $\Pi^0_{jk}=0$) the second theorem follows as a corollary to the first. The second equation of (3.4) is incorrectly written. It should be the same as the second equation of (2.6) in note I.] *P. O. Bell* (Lawrence, Kan.).

Yano, Kentaro. Sur le parallélisme et la concourance dans l'espace de Riemann. Proc. Imp. Acad. Tokyo 19, 189–197 (1943). [MF 14812]

If the holonomic group of a Riemann space leaves a point invariant, it is proved that there exists a vector field v^λ whose members all pass through a fixed point when the space is developed onto its tangent space. Such a vector field is called concurrent. The vector v^λ satisfies the equations $v^\lambda_{;\mu}+\delta_\mu^\lambda=0$. The author examines the properties of concurrent vector fields both in the given Riemann space and in a hypersurface of it. In particular, a necessary and sufficient condition that a space admit a concurrent field is that a coordinate system exist in which its fundamental form has the expression $ds^2=(x^n)^2g_{ij}(x^l)dx^idx^j+dx^ndx^n$,

$i, j=1, \cdots, n-1$. The well-known properties of parallel fields of vectors are derived without reference to the work of other writers. Finally, the author investigates the properties of an Einstein space admitting a parallel or concurrent field of vectors. *C. B. Allendoerfer* (Haverford, Pa.).

Maeda, Jusaku. Differential Möbius geometry of plane curves. Jap. J. Math. **18**, 67–260 (1942). [MF 14968]

The author develops many properties of curves which are invariant under the six-parameter Möbius group of the plane. The natural equation of a curve is $K=K(p)$, where p is the inversive arc length and K is the inversive curvature; this was first obtained by Mullins [Columbia University dissertation, 1917]. Inversive geometry of curves and surfaces in Euclidean n-dimensional space has been studied by Fialkow [Trans. Amer. Math. Soc. **51**, 435–501 (1942); **56**, 309–433 (1944); these Rev. **3**, 307; **6**, 105]. Some of the topics considered are the following. (1) The relative coordinate (which is inversively invariant) of a point with respect to a given curve. (2) The curves associated with a given curve by letting the relative coordinate be constant. (3) The order of contact of two given curves in terms of the inversive invariants. (4) The vertices of first and second order of a given curve. These are the places where the derivatives of first order of the Euclidean and inversive curvatures are zero. (5) The osculating double spirals (K=constant) of a given curve. (6) The analogues of Bouquet's formulas. (7) Homographs and involutions associated with a given curve. (8) Transformations of the relative coordinate. (9) Osculating bicircular quartics of a given curve. (10) Inversion evolutes of a curve. (11) Isothermal systems. (12) Finally, many special curves with interesting inversive properties are studied. *J. DeCicco* (Chicago, Ill.).

Botella Raduan, F. Note on the geometric meaning of a space with affine connection in relation to the ambient affine space. Revista Mat. Hisp.-Amer. (4) **5**, 111–112 (1945). (Spanish) [MF 15202]

The author claims to have developed an interpretation of parallelism in an affinely connected space in terms of parallelism in the tangent affine space. He first assumes metric parallelism in the sense of Levi-Civita and then proceeds to affine parallelism by removing the influence of the metric.
C. B. Allendoerfer (Haverford, Pa.).

Walker, A. G. Completely symmetric spaces. J. London Math. Soc. **19**, 219–226 (1944). [MF 14571]
Walker, A. G. Complete symmetry in flat space. J. London Math. Soc. **19**, 227–229 (1944). [MF 14572]

An n-space is said to have spherical symmetry about a point O if there are coordinates x^i, all zero at O, such that the space admits all orthogonal transformations of x^i, which means that the structure is transformed into itself by every orthogonal transformation. The author considers the effect of complete symmetry about every point upon the structure of generalized spaces. It is proved that a completely symmetric Finsler space is a Riemannian space of constant curvature. A projective path space which is completely symmetric is projectively flat. A completely symmetric affine space is either flat or curved. In the latter case the paths and parameter may be put into correspondence with the geodesics and geodesic arc of a Riemannian space of constant curvature $+1$ or -1. It is pointed out that the same results are obtained if only complete local symmetry is required (with respect to "distances" of the third order). In the second paper the author considers the effect of introducing a unique local angular measure at each point of a completely symmetric affine flat path space in such a way that the complete symmetry is preserved. It is shown that this addition makes the space Euclidean, although without an absolute metric. *J. Haantjes* (Amsterdam).

Cartan, Élie. Sur une classe d'espaces de Weyl. Ann. Sci. École Norm. Sup. (3) **60**, 1–16 (1943). [MF 14636]

The Weyl spaces in question are of three dimensions and admit ∞^2 isotropic planes. The object of the paper is to give a characterization of their curvature tensors and to determine their degree of generality. They are found to depend on four arbitrary functions of two arguments. The author notes that this type of problem is rarely attacked and that his method has wide application. In oversimplified outline the method consists in constructing a Pfaffian system having components of the curvature tensor as independent variables and components of its covariant derivative as unknown functions and applying the known theory of such differential systems to determine the nature of the solutions. An additional result emerges from a consideration of Cauchy's problem for the system: the assignment of a noncharacteristic surface (that is, with nonisotropic tangent planes) in a Weyl space of the above class completely determines the latter. *J. L. Vanderslice* (College Park, Md.).

Laptev, G. Sur l'immersion d'un espace à connexion affine dans un espace affine. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 531–534 (1945). [MF 14416]

Schlaefli's theorem on the immersion of a Riemannian space in a Euclidean space was given a definitive proof by É. Cartan [Ann. Soc. Polonaise Math. **6**, 1–7 (1927)]. Proof of the corresponding theorem for projectively connected spaces was given along the same lines in abbreviated form by S. S. Chern [Bull. Sci. Math. (2) **61**, 234–243 (1937)]. The present paper extends these investigations to affinely connected spaces and follows the Cartan idiom in setting and solving the problem. The fundamental system of differential equations is written in Pfaffian form and the possibility of their solution made to depend on the inductive construction of a sequence of integral elements of ascending dimension terminating at the dimension of the given space. The main difficulties beyond those present in the Riemannian case are resolved with two algebraic lemmas (not proved in this note). The resulting theorem is that every affinely connected space of n dimensions without torsion can be immersed in an ordinary affine space of N dimensions if $N \geqq \frac{1}{2}(n^2+2n-1)$. *J. L. Vanderslice.*

Wagner, V. The theory of geometric objects and the theory of finite and infinite continuous groups of transformations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 347–349 (1945). [MF 13753]

A geometric object of the class v in X_n has a transformation of the form

$$(1) \qquad '\Omega^i = F^i(\Omega^j, \xi^k, f^k(\xi^\lambda), f^k_{\lambda_1\cdots\lambda_s}(\xi^\lambda)), \quad s=1, \cdots, v,$$

where (2) $'\xi^k = f^k(\xi^\lambda)$. It is called a differential geometric object if the functions F^i do not involve ξ^k and f^k explicitly. The equations (1) and (2) together define a group called the total group of the geometric object Ω^i. For $\xi^k=\xi_0{}^k$ in (1) we obtain the local group of Ω^i. Under a point transformation $^*\xi^k=\phi^k(\xi^\lambda)$ every geometric object $\Omega^i(\xi^\lambda)$ is trans-

formed into a geometric object $^*\Omega(^*\xi^k)$ defined by the equation

$$^*\Omega^i(\phi^k(\xi^\lambda)) = F^i(\Omega^j(\xi^\lambda), \phi^k_{\lambda_1\cdots\lambda_s}(\xi^\lambda)).$$

The totality of transformations for which Ω is invariant ($^*\Omega$ identical with Ω) determine a Lie group. From a theorem of Medolaghi it follows that each continuous Lie group is the group of invariance of a certain differential geometric object. A one-parameter family of point transformations belongs to the group of invariance of the geometric object Ω^i if the Lie derivative $(\underset{L}{D}\Omega_i)$ vanishes. *J. Haantjes.*

RELATIVITY

Gomes, Ruy Luis. Les changements de référentiel et la cinématique des ensembles de points. Problèmes qui en dépendent. J. Phys. Radium (8) 1, 335–340 (1940). [MF 13985]

The author considers an ensemble of points E in motion and defines a configuration of the ensemble, with respect to a coordinate system R, to be the positions of these points at some fixed time t. In general, the configuration of points, with respect to another coordinate system R', obtained by means of the Lorentz transformations will not be the corresponding configuration of E with respect to R'. This paper then considers the problem of determining the conditions under which these two configurations will be the same. Various applications are made. *M. Wyman.*

Einstein, Albert. A generalization of the relativistic theory of gravitation. Ann. of Math. (2) 46, 578–584 (1945). [MF 14121]

The author gives a generalization of the relativistic theory of gravitation which is unified in the sense that neither the field equations nor the Hamiltonian function can be expressed as the sum of several invariant parts. The generalization is obtained by assuming that the field is described by a Hermitian symmetric tensor g_{ik} whose values are complex numbers. The existence of a complex linear connection Γ^i_{jk} is also assumed and the concept of parallel displacement is introduced by means of this linear connection. The Hamiltonian function is expressed in terms of g_{ik}, Γ^i_{jk} and their derivatives and by means of a variational principle the field equations are obtained. Although no explicit solutions of the field equations are given, the paper concludes with the remark that the construction of the equations of first approximation shows that they are weaker than Maxwell's equations. *M. Wyman.*

Lichnerowicz, André. Sur une proposition fondamentale de la théorie relativiste de la gravitation. C. R. Acad. Sci. Paris 221, 652–654 (1945). [MF 15151]

The author discusses a problem in relativistic gravitation which has been considered by himself and also by A. Einstein and W. Pauli. The methods used and the results obtained are compared. *M. Wyman* (Edmonton, Alta.).

Lichnerowicz, André. L'intégration des équations de la gravitation relativiste et le problème des n corps. J. Math. Pures Appl. (9) 23, 37–63 (1944). [MF 13796]

It is assumed that the values of the gravitational potentials and their first time derivatives are given on a hypersurface S and various conditions are discussed under which the gravitational equations will have a solution. Conditions for which the solution is unique are also given. The author considers the field equations both for empty space and inside matter. He then makes an application of these results to the problem of n bodies. *M. Wyman.*

Graef Fernández, Carlos. The expansion of the universe in Birkhoff's theory. Bol. Soc. Mat. Mexicana 2, 43–50 (1945). (Spanish) [MF 14479]

In Birkhoff's theory of gravitation, if the pressure p and density ρ of the basic "perfect fluid" are connected by the relation $p=\frac{1}{3}\rho$, then the disturbance-velocity is that of light. But the latter requirement is equally well secured by taking $p=\frac{1}{3}\rho+K\pi^{-1}$, where, in the author's terminology, K is the cosmological constant. This leads to gravitational equations of a type proposed by Barajas, Birkhoff, Graef and Vallarta [Phys. Rev. (2) 66, 138–143 (1944); these Rev. 6, 72]. The author here considers the part of the gravitational potential h_{ij} due to the cosmological term, thereby obtaining the equations of motion of particles acted on by cosmological forces only. He finds that such particles have rectilinear trajectories, and that, if at an initial instant t_0 they are all in the neighbourhood of a given point x_0, then after a very great interval of time their velocities of recession are approximately proportional to their distances from the point. In fact, for motion along the x-axis, $dx/dt=X/T$, approximately, where $X=x-x_0$ and $T=t-t_0$. Thus $1/T$ is Hubble's "constant" and T is the age of the universe. [It may be noted that the author's two conditions on p. 47 are in fact equivalent.] *H. S. Ruse* (Southampton).

Barajas, Alberto. On Einstein's principle of equivalence. Bol. Soc. Mat. Mexicana 2, 51–56 (1945). (Spanish) [MF 14480]

Defining the velocity and acceleration of a particle whose world-line in a general space-time is $x^i=x^i(s)$ to be dx^i/ds and d^2x^i/ds^2, respectively, the author seeks conditions under which the acceleration of a particle acted on by gravitational forces only may be annulled by a change of world-coordinates. He finds that space-time must have an affine connection, and concludes that Einstein's principle of equivalence may be stated in the form that "the world-lines of particles subject only to gravitational forces are paths" and that the principle of equivalence is valid in Birkhoff's theory [same Bol. 1, no. 4 and 5, 1–23 (1944); these Rev. 6, 240]. *H. S. Ruse* (Southampton).

Alba Andrade, Fernando. Gravitational fields of rotating bodies. Bol. Soc. Mat. Mexicana 2, 57–63 (1945). (Spanish) [MF 14481]

In this paper, which is a contribution to Birkhoff's theory [same Bol. 1, no. 4 and 5, 1–23 (1944); these Rev. 6, 240], the author obtains the gravitational potentials h_{ij} for the field of a rotating ring of unit cross section and of constant density. Thence by integration he obtains the h_{ij} at a point exterior to and in the plane of a rotating disc whose density is variable but centrally symmetric. This enables him to calculate the h_{ij} for a rotating sphere of constant density, and to derive differential equations for the motion of a particle in the gravitational field of the sphere. *H. S. Ruse* (Southampton).

Shanmugadhasan, S. On Mathisson's variational equation of relativistic dynamics. Proc. Cambridge Philos. Soc. 42, 54–61 (1946). [MF 14411]

Mathisson [Proc. Cambridge Philos. Soc. 36, 331–350 (1940); these Rev. 2, 207] has found the variational equation of relativistic dynamics for a noncomplete system limited by a world tube to be of the form

$$\int_{s_1}^{s_2} \{X^\alpha \xi_\alpha + (X^{[\alpha,\beta]} + m^{\alpha\beta})\partial_\alpha \xi_\beta + m^{\lambda\alpha\beta}\partial_\lambda \partial_\alpha \xi_\beta + m^{\lambda\kappa\alpha\beta}\partial_\lambda \partial_\kappa \partial_\alpha \xi_\beta + \cdots\} ds = 0,$$

where ξ is an arbitrary vector field vanishing, together with all its derivatives, at the limits s_1 and s_2. In this equation the infinite sequence of m-tensors is characteristic of the physical system while the X's are characteristic of the forces. The equation is said to be solved when the system of relations between the m's, the X's and the path L, compatible with the variation allowed to the ξ's, is found. The present paper considers the solution of the general variational equation using a generalization of the methods of Mathisson. The solution leads to two equations, one for the translation and one for the rotational motion. The last equation is also investigated by setting up a variational equation using the conservation of angular momentum. This is found to be of the same form as that obtained from Mathisson's variational equation. *J. Haantjes.*

de Beauregard, Olivier Costa. Sur la théorie des milieux doués d'une densité de moment cinétique propre. C. R. Acad. Sci. Paris 218, 31–33 (1944). [MF 13448]

The author continues previous work [J. Math. Pures Appl. (9) 21, 267–275 (1942); same C. R. 214, 904–906 (1942); these Rev. 5, 131]. The proportionality of density of matter and energy is assumed and consequences of this assumption are obtained. The density of matter is expressed as a function of the rest-density and velocity. From this expression upper and lower bounds of the density of matter are obtained. The lower bound differs from the classical result. *M. Wyman* (Edmonton, Alta.).

de Beauregard, Olivier Costa. Définition covariante de la force. C. R. Acad. Sci. Paris 221, 743–745 (1945). [MF 15165]

Whittaker, Edmund. Eddington's theory of the constants of nature. Math. Gaz. 29, 137–144 (1945). [MF 13928]

MECHANICS

Elasticity

Green, A. E., and Taylor, G. I. Stress systems in aeolotropic plates. III. Proc. Roy. Soc. London. Ser. A. 184, 181–195 (1945). [MF 13354]

[Parts I and II appeared in Philos. Mag. (7) 34, 416–420, 420–422 (1943); these Rev. 5, 26. Part IV appeared in Proc. Roy. Soc. London. Ser. A. 180, 173–208 (1942); these Rev. 4, 123. Cf. also the three following reviews.] An infinite plate whose material has two directions of elastic symmetry, parallel to the directions of rectangular coordinate axes chosen in the plane of the plate, is subjected to tension. The plate is supposed to contain a circular hole of radius a with center at the origin of the coordinate system. A uniform tension is applied to the plate at infinity, parallel to one of the coordinate axes. If it is assumed that the plate is in the state of generalized plane stress, then the mean stresses are calculable from the stress function χ satisfying the differential equation

$$(\partial^2/\partial x^2 + \alpha_1 \partial^2/\partial y^2)(\partial^2/\partial x^2 + \alpha_2 \partial^2/\partial y^2)\chi = 0,$$

where the positive constants α_i are expressible in terms of four known elastic constants characterizing the material of the plate.

A formal solution of the problem is obtained in the form of an infinite series, and the stresses are expressed in doubly infinite series which fail to converge for certain values of the arguments on the boundary of the hole. However, the series when they do converge can be summed, and it is shown that the resulting closed expressions represent the solution of the problem for all values of the arguments. The paper contains an elaborate numerical discussion of stresses in the neighborhood of the hole for plates made of oak and spruce.

It appears to the reviewer that the method of attack on plane problems of anisotropic elasticity proposed by S. G. Lechnitzky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 1, 77–90 (1937)] could have been applied to this problem with profit. *I. S. Sokolnikoff.*

Green, A. E. Stress systems in isotropic and aeolotropic plates. V. Proc. Roy. Soc. London. Ser. A. 184, 231–252 (1945). [MF 13355]

[Cf. the preceding review.] An infinite isotropic plate in the z-plane contains a hole. If there exists a conformal transformation $z = F(\zeta)$ mapping the region of the z-plane exterior to the hole on the upper half of the ζ-plane in such a way that the boundary of the hole is mapped on the real axis of the ζ-plane, and if the plate is in the state of generalized plane stress, then one can write down the expressions for stresses in terms of three analytic functions $f(z)$, $g(z)$ and $F(\zeta)$, where f and g are functions introduced by Goursat in the general solution of the biharmonic equation for Airy's stress function. The general expressions for stresses are specialized to the cases when the plate is subjected to a uniform tension applied at infinity in the direction of one of the coordinate axes and when a uniform shearing stress is applied at large distances from the hole. Expressions for stresses on the boundary of the hole (including numerical results) are obtained for holes of several interesting shapes (elliptical, rectangular, and square and triangular holes with rounded corners).

The method used to obtain the stress distribution in isotropic plates is generalized to apply to anisotropic plates made of materials with two directions of elastic symmetry parallel to the coordinate axes. The paper contains some numerical results for the distribution of stress on the boundary of an elliptical hole in a spruce plate subjected to tension.

[Reviewer's comment. The theoretical discussion is closely related to the work on isotropic plates by N. I. Muschelišvili [Math. Ann. 107, 282–312 (1932)] and to the work on anisotropic plates by S. G. Lechnitzky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 2, 181–210 (1939)] and V. Morkovin [Quart. Appl. Math. 1,

116–129 (1943); these Rev. 5, 82]. These authors have given rigorous theoretical solutions for the cases when the plate is in the state of plane stress and when it is deflected by transverse loads.] *I. S. Sokolnikoff* (Madison, Wis.).

Green, A. E. Stress systems in aeolotropic plates. VI. Proc. Roy. Soc. London. Ser. A. **184**, 289–300 (1945). [MF 13357]

[Cf. the preceding review.] The paper contains a discussion of the effect of hyperbolic notches in the edges of an orthotropic plate when it is subjected to tension or flexure by couples. The directions of elastic symmetry are parallel to the coordinate axes and the region in the z-plane $(z=x+iy)$ is bounded by two branches of a hyperbola defined by the transformation $z=c\sinh\zeta$, $\zeta=\xi+i\eta$, where the lines $\eta=\pm\alpha$ correspond to the hyperbolic boundaries of the plate. The author makes use of the general expressions for stresses derived in part V [see the preceding review] and considers two cases: (a) plate stretched by a force applied in the direction of the x-axis; (b) plate bent by a couple in the plane of the plate, across a section $x=$constant. The theoretical results are illustrated by a numerical discussion of stresses in plates made of spruce and are used to determine the type of failure which might occur near notches when the plate is in tension. *I. S. Sokolnikoff.*

Green, A. E. Stress systems in aeolotropic plates. VII. Proc. Roy. Soc. London. Ser. A. **184**, 301–345 (1945). [MF 13358]

In parts IV [same Proc. Ser. A. **180**, 173–208 (1942); these Rev. **4**, 123] and V [see the second preceding review], the author considered the effect of unstressed holes on certain distributions of stress in large orthotropic plates. The present paper is concerned with the study of the stress distribution when the displacements and/or stresses are prescribed on the boundary of a hole of a fairly general shape. The attack is that of the papers cited. Part I of the present paper is given to a discussion of a general method of solution of the problem when the plate contains a single hole, on the boundary of which the displacements are specified. Part II is concerned with the distribution of stress in orthotropic plates containing a circular hole when the boundary conditions on the circle are of a mixed variety. Part III deals with the stress distribution in orthotropic plates containing several holes. As one would expect, the numerical work in such problems is quite heavy; the calculations performed so far relate to a spruce plate with two unequal, unstressed circular holes. The plate is assumed to be in tension and the line joining the centers of the holes is perpendicular to the grain. The work is of interest in estimating the effect of bolts and knots in stressed orthotropic plates. *I. S. Sokolnikoff* (Madison, Wis.).

Fox, L., and Southwell, R. V. Relaxation methods applied to engineering problems. VII A. Biharmonic analysis as applied to the flexure and extension of flat elastic plates. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 419–460 (1945). [MF 14286]

[Part VI, by Pellew and Southwell, appeared in Proc. Roy. Soc. London. Ser. A. **175**, 262–290 (1940); these Rev. **2**, 30; cf. the four following titles.] By extension of the technique described in earlier papers, biharmonic analysis and the solution of the equation $\nabla^4 w=W$ are brought within range of the relaxation method.

From the authors' summary.

Christopherson, D. G., Fox, L., Green, J. R., Shaw, F. S., and Southwell, R. V. Relaxation methods applied to engineering problems. VII B. The elastic stability of plane frameworks and of flat plating. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 461–487 (1945). [MF 14287]

Allen, D. N. de G., Fox, L., Motz, H., and Southwell, R. V. Relaxation methods applied to engineering problems. VII C. Free transverse vibrations of membranes, with an application (by analogy) to two-dimensional oscillations in an electro-magnetic system. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 488–500 (1945). [MF 14288]

Allen, D. N. de G., Fox, L., and Southwell, R. V. Relaxation methods applied to engineering problems. VII D. Stress distributions in elastic solids of revolution. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 501–537 (1945). [MF 14289]

Green, J. R., and Southwell, R. V. Relaxation methods applied to engineering problems. VIII A. Problems relating to large transverse displacements of thin elastic plates. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 539–578 (1945). [MF 14290]

Part VIII, by Southwell and Vaisey, part IX, by Green and Southwell, and parts X and XI, by Allen and Southwell, appeared in Proc. Roy. Soc. London. Ser. A. **182**, 129–151 (1943); Philos. Trans. Roy. Soc. London. Ser. A. **239**, 367–386 (1944); Proc. Roy. Soc. London. Ser. A. **183**, 125–134, 258–283 (1944); these Rev. **5**, 161, 247; **6**, 137, 219.

Rosenblatt, Alfred. On the theory of elastic arches. I. Fundamental equations. Actas Acad. Ci. Lima **6**, 121–129 (1943) (Spanish) [MF 14281]

Identical with a paper in Revista Ci., Lima **45**, 241–308 (1943); these Rev. **6**, 81.

Rosenblatt, Alfred. On the theory of the elastic arch. II. New formulas and comparison with the formulas of the North American engineers. Actas Acad. Ci. Lima **6**, 79–110 (1943). (Spanish) [MF 14280]

Continuation of the paper quoted above.

Pickett, Gerald. Flexural vibration of unrestrained cylinders and disks. J. Appl. Phys. **16**, 820–831 (1945). [MF 14541]

The conventional procedures for determining the modes of vibration of slender rods and thin plates involve the use of nonrigorous one-dimensional and two-dimensional analyses, respectively. These treatments imply the omission from the exact equations of terms whose effect becomes large when the breadth and thickness of the bodies become comparable to their length and diameter. The present paper discards the conventional procedure and treats the problems of the cylinder and disc as boundary value problems in the three-dimensional theory of elasticity. The solutions obtained fail to satisfy the boundary condition of vanishing stress at the curved surface of the disc and the ends of the cylinder, but the stress resultants are made to vanish. The error arising from this lack of rigor is evidently small for moderate ratios of diameter to length of the cylinder and length to diameter of the rod. Numerical correction factors which can be applied to the results of the more elementary theories are quoted for various values of the parameters.

G. F. Carrier (Cambridge, Mass.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Cruickshank, F. D. The paraxial differential transfer coefficients of a lens system. J. Opt. Soc. Amer. **36**, 13–19 (1946). [MF 15131]

Another derivation of the transfer coefficient of a lens system within the region of Gaussian optics. [Compare O. N. Clark and S. Rosin, same J. **31**, 394–401 (1941); M. Herzberger, same J. **33**, 651–655 (1943); these Rev. **5**, 112, where the explicit results are given by means of determinants, or Gaussian brackets.] *M. Herzberger.*

Durand, Émile. Étude analytique des systèmes centrés (dioptrique élémentaire). Rev. Optique **23**, 91–104 (1944). [MF 14555]

This paper gives a method of dealing with the paraxial optics of a system of centered surfaces of rotation. Starting from Fermat's principle a set of linear recursion formulae is obtained for the coordinates of intersection of a paraxial ray on the successive surfaces of the system. The coefficients of the recursion formulae involve only the separations, indices and vertex curvatures of the system. Study of the paraxial imagery thus reduces to the solution of a system of s linear equations in s unknowns. The general solution is simplified by use of four subdeterminants in terms of which the customary paraxial characteristics are expressed in a simple manner. The results are used to derive Cotes's formula [cf. J. Southall, Principles and Methods of Geometrical Optics, Macmillan, New York, 1913, pp. 191–197], a generalization of the Lagrange-Helmholtz formula [cf. Southall, ibid., pp. 267–268] and a means of finding the characteristics of a compound optical system from those of its components by matrix multiplication.

A. J. Kavanagh (Buffalo, N. Y.).

Duffieux, P.-Michel. Sur une nouvelle évaluation du stigmatisme approché. C. R. Acad. Sci. Paris **220**, 846–848 (1945). [MF 14071]

The relation between the complex displacements (wave-functions) $F(x, y)$ in the exit-pupil and $G(u, v)$ in the focal plane of an optical system is given, with the usual approximations, by the equation

$$G(u, v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x, y)e^{-2\pi i(ux+vy)}dxdy.$$

Applying the theory of Fourier transforms, the author derives formally three equations, of which the third,

$$(1)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{|\partial F/\partial x|^2+|\partial F/\partial y|^2\}dxdy =(2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u^2+v^2)|G(u, v)|^2dudv,$$

connects the (infinite) moment of inertia about the (u, v)-origin of the energy-distribution $|G(u, v)|^2$ with the function $F(x, y)$, while the first two do the same for the moments of inertia about the u and v axes. The right hand side of (1) is infinite because the contributions from large values of u, v do not fall off sufficiently rapidly; in physical terms, the moment of inertia of the widely diffracted illumination is large. The left hand side is infinite because of the infinite values of $|\partial F/\partial x|^2$ and $|\partial F/\partial y|^2$ at the sharp edge of the exit-pupil.

The author asserts that, since the widely diffracted illumination is known to be associated with diffraction at the boundary of the exit-pupil, the contribution of this illumination to the right hand side of (1) can be separated off from that of the image-aberrations, and that the (finite) integrals

$$\iint|\partial F/\partial x|^2dxdy,\quad \iint|\partial F/\partial y|^2dxdy,$$
$$\iint\{|\partial F/\partial x|^2+|\partial F/\partial y|^2\}dxdy,$$

taken over the interior of the exit-pupil, measure to a sufficient approximation the increase in the three moments of inertia of $|G(u, v)|^2$ caused by the presence of the classical aberrations of geometrical optics. *E. H. Linfoot.*

Duffieux, P.-Michel. Sur une nouvelle évaluation de la distorsion et de la coma. C. R. Acad. Sci. Paris **220**, 911–913 (1945). [MF 14075]

This is a continuation of the note reviewed above. In order that the moments of inertia there defined shall serve as a measure of the dispersal of the luminous image-patch, they must be taken about axes through its centre of gravity (ξ, η). Expressions for ξ and η are derived which, in the particular case $F(x, y)=e^{i\phi(x, y)}$ of uniform intensity over an exit-pupil of area S, reduce to curvilinear integrals

$$\xi=(1/S)\int\phi(x, y)dy,\quad \eta=(1/S)\int\phi(x, y)dx$$

taken around the boundary of the exit-pupil.

The mathematical analysis is vitiated throughout by failure to take account of the divergence of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u|G(u, v)|^2dudv,\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v|G(u, v)|^2dudv$$

and of the corresponding integrals for the moments of inertia about the axes Ou, Ov; no attempt is made to deal with these integrals by physical arguments as in the former note. *E. H. Linfoot* (Bristol).

Barreca, P. Sur la diffraction par un corps noir quelconque de la lumière provenant d'une source punctiforme et équirayonnante. Intégrale complète portée aux quadratures. Arch. Sci. Phys. Nat., Geneva **22**, 234–261 (1940). [MF 14189]

This is a continuation of earlier work by the author [same Arch. **18**, 88–99 (1936); **19**, 237–265 (1937)]. The following is the author's summary. "Dans le présent travail on obtient l'intégrale complète de l'équation indéfinie de l'optique pour le cas exposé dans le titre et on indique brièvement la façon d'en déduire, peut-être et dans des cas particuliers simples, l'intégrale générale lorsqu'on aura put exécuter la quadrature. Nous appelons complètes (suivant J.-L. Lagrange) les intégrales générales lorsque leurs fonctions arbitraires dégénèrent en constantes (arbitraires). La surface du corps est supposée avoir une équation unique. Le problème considéré n'avait pas été encore résolu."

Unfortunately it is practically impossible to follow the author's proofs and so to judge whether the claim is justified. It seems likely that it is not, for two reasons. (i) The author nowhere states the boundary conditions satisfied at the surface of the black body. (ii) The work is based on Whittaker's solution

$$V=\int_{-\pi}^{\pi}\int_{0}^{\pi}\exp\{ik(x\sin u\cos v +y\sin u\sin v+z\cos u)\}f(u, v)dudv$$

of the equation $\nabla^2 V + k^2 V = 0$, and this solution cannot represent expanding spherical waves. [Cf. Weyl, Ann. Physik (4) 60, 481–500 (1919).] *E. T. Copson* (Dundee).

Würschmidt, José. Aberration, Doppler effect and pressure of light. Revista Unión Mat. Argentina 11, 47–68 (1945). (Spanish. English summary) [MF 14665]

Hammad, A. The primary and secondary scattering of sunlight in a plane-stratified atmosphere of uniform composition. II. Numerical tables and discussion of the directional distribution of the primary scattered light. Philos. Mag. (7) 36, 434–440 (1945). [MF 15481]

This sequel to an earlier theoretical paper [Philos. Mag. (7) 28, 99–110 (1939)] gives some calculated values of the primary scattered solar radiation in various directions with respect to the sun. The brief tables are confined to zenith angles 0° and 60° and to one value of the total extinction coefficient. *W. E. K. Middleton* (Toronto, Ont.).

Raymond, François. Remarques sur les coordonnées symétriques de Fortescue. C. R. Acad. Sci. Paris 218, 113–115 (1944). [MF 13460]

This note is concerned with a discussion of the symmetric coordinates of Fortescue, defined in electrotechnics by the following matrix α_i^μ, in terms of coordinate changes in unitary geometry:

$$(1) \qquad (\alpha_i^\mu) = (a^{-(i-1)(\mu-1)}),$$

where a is a root of unity. The total (complex) energy H of an n-phase network is given by (2) $2H = Z^{ij}\bar{x}_i x_j$, where $\bar{x}$ is the conjugate complex of x, and (3) $u^i = \partial H/\partial \bar{x}_i$, where x_i is the sinusoidal current and u^i is the potential in the ith branch. The problem is to find new variables X_μ, where (4) $x_i = \alpha_i^\mu X_\mu$, such that (a) the quadratic form (2) becomes diagonal and (b) the potentials, computed by (3), transform contragrediently to (4). It is shown that the Fortescue solution (1) is a special case of a more general solution $(\alpha_i^\mu) = (n^{-\frac{1}{2}} e^{j\theta} a^{-(i-1)(\mu-1)})$, $j^2 = -1$. *A. L. Foster.*

Liénard, Alfred. Application de la thermodynamique aux théories électrodynamiques de Hertz et de H. Lorentz pour les corps en mouvement. Ann. Fac. Sci. Univ. Toulouse (4) 5, 1–48 (1941). [MF 14691]

This paper is interesting in that it suggests a method of developing the properties of the electromagnetic field by thermodynamical arguments. The method was first developed in an earlier paper of the author [Ann. Physique (9) 20, 249–360 (1923); see also A. O'Rahilly, Electromagnetics, Longmans, London; Cork University Press, 1938, pp. 126–151]. The work is extended in the present paper to take into account the finite velocity of propagation of electromagnetic disturbances. Application to Lorentz's theory is given in detail. It is shown that the density of energy flux is given by

$$\mathfrak{T} = \mathbf{E}\times\mathbf{H} - \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\mathbf{V} - \mathbf{B}\cdot\mathbf{H}\mathbf{V} - \mathbf{V}\cdot\mathbf{J}\mathbf{B},$$

where $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{B}$ have the usual significance, $\mathbf{V}$ is the velocity of a charged particle and $\mathbf{J}$ the intensity of magnetization. It is seen that the above expression reduces to the well-known Poynting vector $(\mathbf{E}\times\mathbf{H})$ if $\mathbf{V} = \mathbf{J} = 0$.

In the last of the five chapters, the author considers the application of his theory to thermoelectric effects. He departs from the usual procedure [such as that of P. W. Bridgman, The Thermodynamics of Electrical Phenomena in Metals, Macmillan, New York, 1934, pp. 49ff.] by considering the expression $dQ/T - dS = du$, where du is so chosen that the volume integral of du/dt vanishes; that is, $du = \operatorname{div}(\theta i)d\tau dt$. Here i is the conduction current density and θ is the thermoelectric power. It is shown that the expressions for heat absorption and intrinsic electromotive force are essentially in agreement with those obtained earlier by W. Thomson [Mathematical and Physical Papers, vol. 1, Cambridge University Press, 1882, pp. 174–291].
C. Kikuchi (East Lansing, Mich.).

Guggenheim, E. A. Note on magnetic energy. Phys. Rev. (2) 68, 273–276 (1945). [MF 14667]

In classical electrodynamics, one uncritically assumes that the contribution of the magnetic field to the energy density is given by $\mu H^2/(8\pi)$. This assumption has been examined and criticized in an earlier paper of the author [Proc. Roy. Soc. London. Ser. A. 155, 49–70, 70–101 (1936)] and recently by G. H. Livens [Philos. Mag. (7) 36, 1–20 (1945); these Rev. 6, 283]. In the present paper the author considers the contribution of permanent magnetization to the Lagrangian and the Hamiltonian of the field.
C. Kikuchi (East Lansing, Mich.).

Roy, Louis. Complément au mémoire sur les actions magnétiques, électriques, électrodynamiques et électromagnétiques dans les corps rigides ou déformables. Ann. Fac. Sci. Univ. Toulouse (4) 4, 119–148 (1940). [MF 15169]

The first part appeared in the same Ann. (4) 3, 1–69 (1939).

Theodorchik, K. A non-linear theory of RC-generators. Acad. Sci. USSR. J. Phys. 9, 341–345 (1945). [MF 14520]

Using various approximating techniques an analysis of the operation of single and multiphase RC generators is given which does not assume linearity. The resulting theory offers an explanation of a number of familiar phenomena not explainable by linear theory. *N. Levinson.*

Craggs, J. W., and Tranter, C. J. The capacity of twin cable. II. Quart. Appl. Math. 3, 380–383 (1946). [MF 14529]

In a recent paper [same Quart. 3, 268–272 (1945); these Rev. 7, 98] we have given a method for determining the capacity of two circular wires surrounded by concentric touching dielectric sheaths. The present note gives the extension of the method to the case in which the dielectric sheaths are not in contact. *Extract from the paper.*

Fock, V. Diffraction of radio waves around the earth's surface. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 15, 479–496 (1945). (Russian. English summary) [MF 15383]

Fock, V. Diffraction of radio waves around the earth's surface. Acad. Sci. USSR. J. Phys. 9, 255–266 (1945). [MF 14518]

Detailed proofs of results announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 310–313 (1945); these Rev. 7, 100.

Grünberg, G. Theory of the coastal refraction of electromagnetic waves. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 14, 84–111 (1944). (Russian) [MF 12222]

The object of this paper is to develop a theory of the coastal refraction of electromagnetic waves, and of related phenomena. Approximating boundary value conditions are formulated for the electric field component normal to the surface of the earth, and on this basis an integral equation is set up for this component. An approximate solution of

this integral equation is derived and the case of a flat wave overrunning an infinite straight coast line is investigated in detail. Some general discussions are given in regard to the solution of the integral equation for the cases where the waves overrun islands of arbitrary shapes. An appendix is devoted to the evaluation of various integrals involving Bessel functions. These integrals are generalizations of those occurring in the main body of the paper. The author states that some of these results appear to be new.

H. P. Thielman (Ames, Iowa).

Schelkunoff, S. A. Concerning Hallén's integral equation for cylindrical antennas. Proc. I. R. E. **33**, 872–878 (1945).

There is a discrepancy between the values of the impedance of cylindrical antennas as calculated from Schelkunoff's formula [Proc. I. R. E. **29**, 493–521 (1941)] on the one hand and from Hallén's theory [Nova Acta Soc. Sci. Upsaliensis (4) **11**, 1–44 (1938)] on the other. L. Brillouin suggested [Quart. Appl. Math. **1**, 201–214 (1943); these Rev. **5**, 163] that Hallén's approximate integral equation may be at fault. The author derives the exact integral equation for a hollow cylindrical antenna and shows that Hallén's equation is a valid approximation for thin and long cylinders. On the other hand he shows that Hallén's solution depends on a further tacit assumption which is justified only when the antenna is short compared with the wave length. It is on account of this circumstance that Hallén's numerical results do not agree with experimental results in most practically important cases. This view of the position is confirmed by Gray's solution of Hallén's equation [J. Appl. Phys. **15**, 61–65 (1944); these Rev. **6**, 282], which yields results in good agreement with experimental curves and also with Schelkunoff's formula. *A. Erdélyi.*

Harrison, Charles W., Jr. Symmetrical antenna arrays. Proc. I. R. E. **33**, 892–896 (1945).

A relatively simple method is presented for calculating the impedance properties of antenna arrays consisting of n identical radiators oriented at the vertices of regular polygons. All antennas are required to carry currents of equal magnitude, but not necessarily of the same phase. However, the choice of phase angle must be such that the required electrical symmetry of the array is unimpaired.

From the author's summary.

Lüdi, F. Eigenfrequenzen des E-Typus eines kapazitätsbelasteten zylindrischen Hohlraumes. Helvetica Phys. Acta **17**, 429–436 (1944).

The author considers a cavity resonator in the shape of a finite cylindrical annulus. The inner wall is composite, being essentially a cylindrical condenser with a small gap. As an approximation it is assumed that the annulus is infinite in length so that the problem is reduced to one of cylindrical symmetry. Both the solution and boundary conditions are expressed in terms of the z component of the electric field and are: $E_z=0$, $r=b$; $E_z=(2\pi\epsilon/Cl)\int E_z r dr$. This gives rise to the following two transcendental equations determining the resonant frequency:

$$A\{J_0(ka)-(2\pi\epsilon a/Ckl)J_1(ka)\}+B\{N_0(ka)-(2\pi\epsilon a/Ckl)N_1(ka)\}=0,$$

$$AJ_0(kb)+BN_0(kb)=0.$$

The symbols a and b denote the inner and outer radius of the annulus, k is 2π divided by the wave length of the radiation, C is the capacity of the condenser and l is the side of the annulus. These equations are solved numerically for one case and approximate solutions are obtained for the cases where b/a is either small or big. The conductivity losses are also calculated. *H. Feshbach.*

Kahan, Théo. Effet d'un faisceau électronique sur les fréquences propres d'une cavité électromagnétique. C. R. Acad. Sci. Paris **221**, 616–618 (1945). [MF 15150]

Quantum Mechanics

Datzeff, A. Interprétation quantique de certains résultats en mécanique classique. J. Phys. Radium (8) **1**, 368–373 (1940). [MF 13986]

The author proposes a mechanical model of the hydrogen atom. Although it is based on classical dynamics only, remarkable similarity with the Bohr theory is achieved. Essentially, it is assumed that the interaction between proton and electron consists of a small periodic perturbation in addition to the usual Coulomb and gravitational forces. This interaction is obtained by the following mechanical device: the electron is replaced by a pair of particles, one of which carries the electronic charge. The two particles are close together as compared to their distance from the atomic nucleus and all three are assumed to be coplanar.

Neglecting small terms in the potential energy, the system reduces to two two-body problems. The second order term, which is periodic, is then introduced as a small perturbation and the stability of the motion is examined. The stable orbits are exactly those of the Bohr atom if a certain constant is identified with Planck's constant h. The perturbation induces a variation of the action integral, approximately given by sinusoidal functions which are analogous to the de Broglie wave associated with the electron.

A. Schild (Toronto, Ont.).

Courtel, Robert. Perturbation des problèmes de valeurs propres par modification de la frontière: cas des équations de la mécanique ondulatoire. C. R. Acad. Sci. Paris **219**, 311–314 (1944). [MF 15262]

de Beauregard, Olivier Costa. Sur l'invariance de jauge des tenseurs de la théorie de Dirac. Sur l'interprétation d'une formule de Tetrode et d'une formule de M. E. Durand. C. R. Acad. Sci. Paris **218**, 961–963 (1944). [MF 15245]

Bass, Jean. Sur les rapports entre la mécanique ondulatoire et la théorie des fonctions aléatoires. C. R. Acad. Sci. Paris **221**, 46–49 (1945). [MF 14674]

Arnous, Edmond. Sur les fondements de la méthode de la fonction caractéristique quantique [À propos d'une note de M. Jean Bass]. C. R. Acad. Sci. Paris **221**, 489–491 (1945). [MF 14690]

Arnous, Edmond. La fonction caractéristique quantique à plusieurs variables et les spectres des grandeurs simultanément mesurables. C. R. Acad. Sci. Paris **219**, 357–359 (1944). [MF 15265]

Arnous, Edmond. Conditions nécessaires et suffisantes pour que deux grandeurs simultanément mesurables soient stochastiquement indépendantes, pour un état donné du système mécanique. C. R. Acad. Sci. Paris **219**, 389–391 (1944). [MF 15272]

Arnous, Edmond. Étude de la statistique attachée à l'opérateur $i\partial/\partial q$ à l'aide de sa fonction caractéristique. C. R. Séances Soc. Phys. Hist. Nat. Genève 62, 64–66 (1945). [MF 14560]

Arnous, Edmond. Les caractéristiques, selon Cauchy, des équations d'évolution des particules à spin et l'équation relativiste de Jacobi. C. R. Acad. Sci. Paris 219, 672–673 (1944). [MF 15302]

Bhabha, H. J. Relativistic wave equations for the elementary particles. Rev. Modern Phys. 17, 200–216 (1945). [MF 13691]

The question of Lorentz invariant equations of the form

$$(1)\qquad \sum_{k=0}^{3} (p_k\alpha^k+\chi)\psi=0$$

is investigated in some detail. (In these equations, p_k is the differential operator $i\partial/\partial x$, α^k and χ are matrices and ψ is a wave function with several components, the number of components being equal to the number of columns in the α^k and χ.) The investigation differs from those previously given [Majorana, Dirac, Proca, Kemmer, Pauli and Fierz] mainly by the restriction of equations to the form above in which the α^k are square matrices. As a result, no initial conditions similar to those about the divergence of the electric and magnetic field vectors in Maxwell's equations can be introduced. It then follows from an argument already given by Dirac that only one irreducible equation of the above form exists. That equation refers to an ordinary electron, with a spin one-half. The cases with higher spin yield reducible equations in which the mass, that is, the rest energy of the particles, can assume several values [cf. also O. Klein, Ark. Mat. Astr. Fys. 25A, no. 15 (1936)].

Under the Lorentz transformation, ψ goes over into $S\psi$, the matrices S forming a representation of the Lorentz group. Special equations of the form (1) are obtained by extending the representation S of the four-dimensional Lorentz group to a representation of the five-dimensional group. It is shown that α^k can be set equal to the infinitesimal operators which correspond to rotations in the planes formed by the new axis (4) and one of the old axes (0, 1, 2, 3). Since the representations of the five-dimensional Lorentz group (in a finite number of dimensions) can be characterized by two numbers $\lambda_1 \geqq \lambda_2 \geqq 0$, both integers or both half integers, the special kind of equations (1) also can be characterized by two such numbers; λ_1 is called "the spin" of the particle. Since, except for $\lambda_1=0$ and $\lambda_1=\frac{1}{2}$, λ_2 can assume several values for a given λ_1, the spin does not completely determine the nature of the particle even in the special case first considered. It is proved that the rest-mass in the equations obtained in this way can assume any of the values $\pm 2\chi, \pm 2\chi/3, \cdots, \pm 2\chi/(2\lambda+1)$ if the particle has a half integral spin and the values $\pm\chi, \pm\chi/2, \cdots, \pm\chi/\lambda$ if the spin is integral. It is also shown that the equations obtained in this way are only special cases of the more general equations which are also obtained in the second part of the paper. It is suggested that the proton is a low rest-mass state of a particle with a higher spin.

The greater complexity of the results, as compared to those obtained by previous investigators, is due to the restriction mentioned above, that is, the absence of initial conditions. As a consequence of this, practically all of the equations derived would be regarded as reducible in most of the previous work. The appearance of initial conditions in Maxwell's equations indicates that these are, perhaps, not quite so unnatural as the author suspects.

E. P. Wigner (Princeton, N. J.).

McConnell, James. Production and annihilation of negative protons. Proc. Roy. Irish Acad. Sect. A. 50, 189–221 (1945). [MF 14284]

If the wave-functions of the proton and of the neutron obey the Dirac equation then both particles must possess anti-particles. The anti-particle of the proton must be assumed to be the negative proton. The production of proton pairs, consisting of a negative and a positive proton, is considered. As a first step the production of proton pairs in the collision of two oppositely charged pseudo-scalar mesons is treated. The production of proton pairs in the collision of a meson with a nucleon is then treated as the collision between the incident meson and a virtual meson emitted by the nucleon. The cross-section for the production of proton pairs by mesons becomes appreciable for meson energies exceeding $4Mc^2$ (M, the proton mass), is of the order of 10^{-28} cm^2, and decreases slightly with increasing energy. It is estimated that the process should account for about a tenth of the cosmic ray protons observed near sea-level. Though no negative protons have been identified experimentally, the predicted number of negative protons is not in contradiction with the observational evidence. The mean life of the negative proton, in dense materials, is estimated to be of the order of 10^{-4} sec. *L. Jánossy.*

Destouches, Jean-Louis. Sur la théorie du nucléon; le spin isotopique et les forces nucléaires. C. R. Acad. Sci. Paris 219, 575–577 (1944). [MF 15291]

van Isacker, Jacques. Sur les décompositions des équations de particules à spin quelconque. C. R. Acad. Sci. Paris 219, 51–53 (1944). [MF 14507]

Kwal, Bernard. Les représentations spinorielles, supérieures, du groupe de Lorentz et la théorie des corpuscules de masse et de spin multiples. C. R. Acad. Sci. Paris 221, 658–659 (1945). [MF 15154]

Murard, Robert. Définition de quelques types de corpuscules; application au nucléon. C. R. Acad. Sci. Paris 219, 577–579 (1944). [MF 15292]

Murard, Robert. Comportement des corpuscules dans un champ extérieur; application au nucléon. C. R. Acad. Sci. Paris 221, 547–549 (1945). [MF 15144]

Murard, Robert. Sur les divers types de corpuscules élémentaires. C. R. Acad. Sci. Paris 221, 607–609 (1945). [MF 15148]

Petiau, Gérard. Sur certains types d'interactions corpusculaires. C. R. Acad. Sci. Paris 219, 309–311 (1944). [MF 15261]

Petiau, Gérard. Sur les corpuscules de spin 0. C. R. Acad. Sci. Paris 219, 441–443 (1944). [MF 15277]

Slansky, Serge. Propriétés de l'opérateur d'interaction de deux corpuscules de spin 1/2 en mécanique ondulatoire relativiste. C. R. Acad. Sci. Paris 218, 704–706 (1944). [MF 15324]

de Broglie, Louis, et Tonnelat, Marie-Antoinette. Remarques sur quelques difficultés de la théorie du photon, liées à l'emploi d'une solution d'annihilation. C. R. Acad. Sci. Paris 218, 889–892 (1944). [MF 15235]

Bloch, Léon. Sur la relation de Uhlenbeck et Laporte pour le photon. C. R. Acad. Sci. Paris 219, 674–675 (1944). [MF 15303]

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